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Claw-free graphs with strongly perfect complements. Fractional and integral version, Part II: Nontrivial strip-structures

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ABSTRACT

Strongly perfect graphs have been studied by several authors (e.g., Berge and Duchet (1984) [1], Ravindra (1984) [7] and Wang (2006) [8]). In a series of two papers, the current paper being the second one, we investigate a fractional relaxation of strong perfection. Motivated by a wireless networking problem, we consider claw-free graphs that are fractionally strongly perfect in the complement. We obtain a forbidden induced subgraph characterization and display graph-theoretic properties of such graphs. It turns out that the forbidden induced subgraphs that characterize claw-free graphs that are fractionally strongly perfect in the complement are precisely the cycle of length 6, all cycles of length at least 8, four particular graphs, and a collection of graphs that are constructed by taking two graphs, each a copy of one of three particular graphs, and joining them in a certain way by a path of arbitrary length. Wang (2006) [8] gave a characterization of strongly perfect claw-free graphs. As a corollary of the results in this paper, we obtain a characterization of claw-free graphs whose complements are strongly perfect.

1. Introduction

All graphs in this paper are finite and simple. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and edges, respectively, of G. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise nonadjacent vertices. The *clique number* $\omega(G)$ denotes the size of a maximum cardinality clique in G and the *stability number* $\alpha(G)$ denotes the size of a maximum cardinality stable set in G. Let $\chi(G)$ denote the *chromatic number* of G. We denote by G^c the *complement of* G. We say that a clique K is a *dominant clique in* G if every maximal (under inclusion) stable set S in G satisfies $S \cap K \neq \emptyset$. For another graph H, we say that G *contains* H *as an induced subgraph* if G has an induced subgraph that is isomorphic to H. The *claw* is a graph with vertex set $\{a_0, a_1, a_2, a_3\}$ and edge set $\{a_0a_1, a_0a_2, a_0a_3\}$. We say that a graph G is *claw-free* if G does not contain the claw as an induced subgraph. We say that G is *connected* if there exists a path between every two $u, v \in V(G)$. A *connected component* of G is a maximal connected subgraph of G. For disjoint sets $A, B \in V(G)$, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and $a \in V(G)$ is complete to B if $\{a\}$ is complete to B.

A graph G is perfect if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. We are interested in the following concept.

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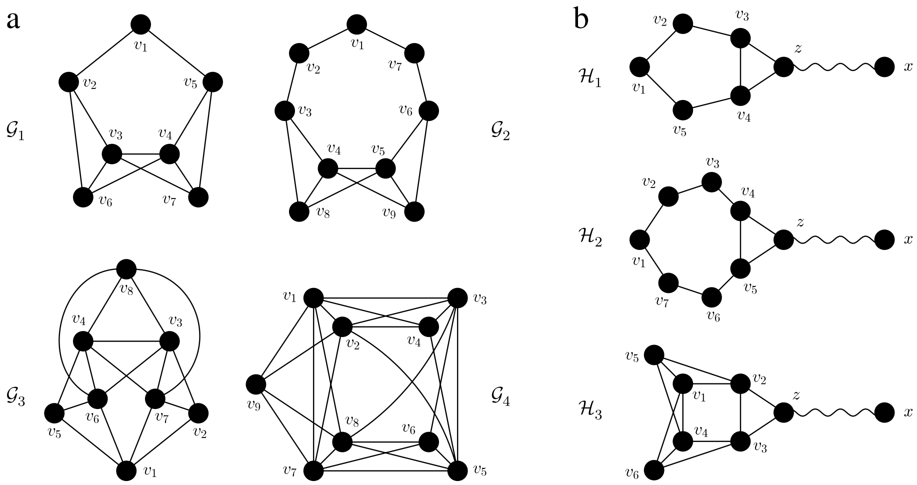


Fig. 1. Forbidden induced subgraphs for fractionally co-strongly perfect graphs. (a) The graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$. (b) Hefts $\mathcal{H}$ that are combined to construct skipping ropes.

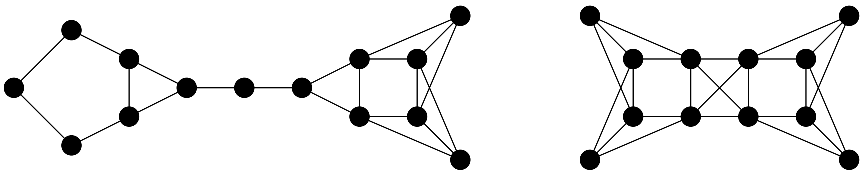


Fig. 2. Two examples of skipping ropes. Left: the skipping rope of type $(1, 3)$ of length 3. Right: the skipping rope of type $(3, 3)$ of length 0.

Definition. A graph G is *fractionally co-strongly perfect* if and only if, for every induced subgraph H of G, there exists a function $w : V(H) \to [0, 1]$ such that

$$\sum_{v \in S} w(v) = 1, \quad \text{for every maximal stable set } S \text{ of } H. \tag{1}$$

We call a function w that satisfies (1) a *saturating vertex weighting for* H.

Next, let us define the following three classes of graphs:

- $\mathcal{F}_1 = \{C_k \mid k = 6 \text{ or } k \geq 8\}$, where C_k is a cycle of length k;
- $\mathcal{F}_2 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$, where the $\mathcal{G}_i$'s are the graphs drawn in Fig. 1(a);
- Let $\mathcal{H} = \{\mathcal{H}_1(k), \mathcal{H}_2(k), \mathcal{H}_3(k) \mid k \geq 0\}$, where $\mathcal{H}_i(k)$ is the graph $\mathcal{H}_i$ drawn in Fig. 1(b) but whose 'wiggly' edge joining z and x is replaced by an induced k-edge-path. For $i \in \{1, 2, 3\}$, we call $\mathcal{H}_i(k)$ a *heft of type i with a rope of length k*. We call x the *end of the heft* $\mathcal{H}_i(k)$.

  Now let $i_1, i_2 \in \{1, 2, 3\}$ and let $k_1, k_2 \geq 0$ be integers. Let $H_1 = \mathcal{H}_{i_1}(k_1)$ and $H_2 = \mathcal{H}_{i_2}(k_2)$, and let x_1, x_2 be the end of heft H_1, H_2, respectively. Construct H from the disjoint union of H_1 and H_2 by deleting x_1 and x_2, and making the neighbors of x_1 complete to the neighbors of x_2. Then H is called a *skipping rope of type* (i_1, i_2) *of length* $k_1 + k_2$. Let $\mathcal{F}_3$ be the collection of skipping ropes. Fig. 2 shows two examples of skipping ropes.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. A graph G is $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{F}$. We say that a graph G is *resolved* if at least one of the following is true:

(a) there exists $x \in V(G)$ that is complete to $V(G) \setminus \{x\}$; or
(b) G has a dominant clique; or
(c) G is not perfect and there exists $k \in \{2, 3\}$ such that every maximal stable set in G has size k.

We say that a graph G is *perfectly resolved* if every *connected* induced subgraph of G is resolved. In a series of two papers (the current paper and [3]), we will prove the following theorem:

Theorem 1.1. *Let G be a claw-free graph. Then the following statements are equivalent:*

(i) *G is fractionally co-strongly perfect;*
(ii) *G is $\mathcal{F}$-free;*
(iii) *G is perfectly resolved.*

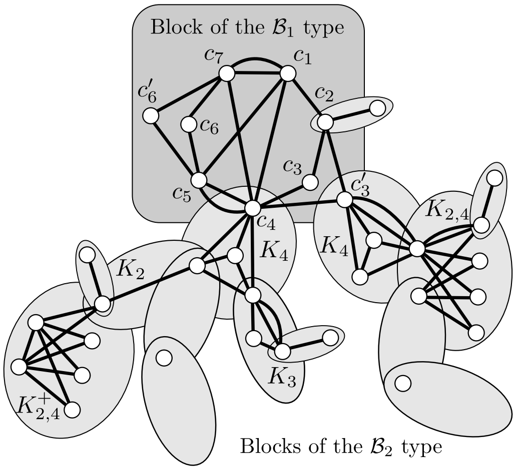


Fig. 3. An example of the pattern multigraph H of an optimal representation (T, H, η) of an $\mathcal{F}$-free claw-free trigraph. The ellipses show the blocks of the multigraph. The pendant edges represent strips (J, Z) that satisfy $1 \leq |Z| \leq 2$. All other edges represent strips (J, Z) with $|Z| = 2$.

Chudnovsky and Seymour [5] proved a structure theorem for claw-free graphs. The theorem roughly states that every claw-free graph is either of a certain 'basic' type or admits a so-called 'strip-structure'. The current paper deals with the proof of Theorem 1.1 for the case when G admits a 'strip-structure'. In fact, [5] deals with slightly more general objects called 'claw-free trigraphs'. What is actually meant by 'a trigraph admits a strip-structure' will be explained in Section 2. The goal of this paper is to prove the following result.

Theorem 1.2. *Every connected $\mathcal{F}$-free claw-free trigraph that is not basic is resolved.*

Theorem 1.2 finishes the proof of the main result of [3] and the current paper.

1.1. *Informal overview of the paper*

The claw-free graphs that we will be dealing with in this paper are graphs that admit so-called strip-structures. Such claw-free graphs are generalizations of line graphs in the following sense. Let H be a multigraph. Think of constructing the line graph G of H in the following way. For every edge e of H, there is a (unique) vertex in G and this vertex is adjacent to all vertices in G that correspond to edges that share an endpoint with e. We can think of H as the 'pattern multigraph' for its line graph. A strip-structure is a generalization of this construction in the following sense. We again start with a multigraph H which we call the pattern multigraph for the strip-structure. In this case, however, for every edge e of H there is a corresponding claw-free graph G_e (an induced subgraph of G) which is either just a vertex (in the same manner as with line graphs), or a so-called 'strip'. Each such a strip G_e is a claw-free graph that contains two special disjoint cliques that are called the 'endcliques', and each endclique corresponds to one endpoint of e. The union of all endcliques corresponding to a specific vertex in the strip-structure is a clique. It turns out (see Section 2.3) that there are fifteen types of strips.

It may happen that there exist multiple strip-structures that describe a fixed claw-free graph. We will always insist on choosing a strip-structure with a maximum number of edges in the pattern multigraph. We call such a strip-structure an optimal strip-structure. The fact that our claw-free graphs are $\mathcal{F}$-free implies that they do not contain long induced cycles (where 'long' means of length six or at least eight). This has particular consequences for the structure of the pattern multigraph for the strip-structure, to be precise for its block decomposition. This structure is investigated in Section 3 (see also Fig. 3 for a preview). We will also be able to prove some results about the lengths of induced paths between endcliques inside strips.

In Section 5, we will start with the proof of Theorem 1.2. This section deals with graphs with stability number at most three and takes a few pages. The bulk of the work is done in Section 6, in which we prove Theorem 1.2 for graphs with stability number at least four. Let G be such a claw-free graph and let H be the pattern multigraph for the optimal strip-structure corresponding to G. We will look at the maximal 2-connected subgraphs of H (*i.e.*, the block-decomposition of H). An induced subgraph of G that corresponds to a maximal 2-connected subgraph of H is called a strip-block. It will turn out that excluding skipping ropes buys us a useful property: at most one 'special' strip-block of G contains an induced cycle of length at least five (but not six). We will call all other strip-blocks 'ordinary'. Thus, unless H is 2-connected (in which case there is only one strip-block), we can always find an ordinary strip-block. Ordinary strip-blocks are relatively simple because of the absence of induced cycle of length at least five and it will turn out that we are always able to find a dominant clique in some ordinary strip-block. Thus, what remains to be considered is the case when H is 2-connected. For this reason, Section 6 is divided into two parts: a part for 2-connected strip-structures (Section 6.2) and a part for non-2-connected strip-structures (Section 6.3).

So far we stated everything in terms of graphs. However, the structure theorem for claw-free graphs of Chudnovsky and Seymour (which is presented in Section 2.3) is stated in terms of more general objects called 'trigraphs'. Trigraphs are like graphs, except that some adjacencies are 'undecided'. Pairs of vertices between which the adjacency is undecided are said to be 'semiadjacent' and in the setting of claw-free graphs the undecided pairs always form a matching. Although all the results in this paper can be stated in terms of graphs, the analysis is considerably easier when stated in terms of trigraphs. The reason for this is the fact that every claw-free graph can be constructed from a trigraph without adjacent clones (see Section 2.1 for a definition) by a 'thickening' operation. This thickening operation blows up each vertex of the trigraph to a clique, replacing edges of the trigraph by complete bipartite graphs, semiedges by arbitrary bipartite graphs containing at least one edge and one nonedge, and nonedges by empty bipartite graphs. At almost all times, we can conclude that this 'thickened' graph is resolved by just looking at the (simpler) trigraph, thereby circumventing an extra layer of complexity. Unfortunately, at a few places in the proof, this is not the case and we have to investigate the thickened graph. This will happen in particular in Section 6.3, because it may not always be possible to determine just by looking at the trigraph which blocks of the strip-structure are ordinary.

1.2. Organization of the paper

This paper is structured as follows. In Section 2, we introduce tools that we need throughout the paper. In the same section, we also present the relevant parts of the structure theorem for claw-free graphs of Chudnovsky and Seymour [5]. This structure theorem is stated in terms of trigraphs, a generalization of graphs, which are also defined in Section 2. Then, in Section 3, we will present the proof of Theorem 1.1 assuming the validity of Theorem 1.2. In Section 3, we will start with a structure theorem for the pattern multigraph for the strip-structure of $\mathcal{F}$-free graphs. Section 5 deals with $\mathcal{F}$-free claw-free graphs that are not basic and that have stability number at most three. Finally, in Section 6, we prove Theorem 1.2 for the remaining claw-free graphs.

2. Tools

In this section, we introduce definitions, notation and important lemmas that we use throughout the paper. As in [5], it will be helpful to work with "trigraphs" rather than with graphs. We would like to point out that the results in [5] can be stated in terms of graphs as well. Although we originally tried to write this paper using the graph-versions of these results, we quickly realized that whether a graph is resolved can – up to a few exceptions – easily be determined from the underlying trigraph. Therefore, working with trigraphs rather than their graphic thickenings (see Section 2.1) simplifies the analysis considerably. We use the terminology defined in this section for graphs as well. The definitions should be applied to graphs by regarding graphs as trigraphs. We next state some results from [3], the proofs of which we omit here. They can be found in [3].

2.1. Claw-free graphs and trigraphs

For an integer $n \geq 1$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. In this section we define terminology for trigraphs. We use this terminology defined for trigraphs in this section for graphs as well. The definitions should be applied to graphs by regarding graphs as trigraphs.

A *trigraph* T consists of a finite set $V(T)$ of vertices, and a map $\theta_T : V(T) \times V(T) \to \{1, 0, -1\}$, satisfying:

- $\theta_T(v, v) = 0$, for all $v \in V(T)$;
- $\theta_T(u, v) = \theta_T(v, u)$, for all distinct $u, v \in V(T)$;
- for all distinct $u, v, w \in V(T)$, at most one of $\theta_T(u, v), \theta_T(u, w)$ equals zero.

We call θ_T the *adjacency function* of T. For distinct $u, v \in V(T)$, we say that u and v are *strongly adjacent* if $\theta_T(u, v) = 1$, *strongly antiadjacent* if $\theta_T(u, v) = -1$, and *semiadjacent* if $\theta_T(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. We denote by $F(T)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(T)$ are distinct and semiadjacent. Thus a trigraph T is a graph if $F(T) = \emptyset$.

We say that u is a *(strong) neighbor* of v if u and v are (strongly) adjacent; u is a *(strong) antineighbor* of v if u and v are (strongly) antiadjacent. For distinct $u, v \in V(T)$ we say that $uv = \{u, v\}$ is an *edge*, a *strong edge*, an *antiedge*, a *strong antiedge*, or a *semiedge* if u and v are adjacent, strongly adjacent, antiadjacent, strongly antiadjacent, or semiadjacent, respectively. For disjoint sets $A, B \subseteq V(T)$, we say that A is *(strongly) complete to* B if every vertex in A is (strongly) adjacent to every vertex in B, and that A is *(strongly) anticomplete to* B if every vertex in A is (strongly) antiadjacent to every vertex in B. We say that A and B are *linked* if every vertex in A has a neighbor in B and every vertex in B has a neighbor in A. For $v \in V(T)$, let $N_T(v)$ denote the set of vertices adjacent to v, and let $N_T[v] = N_T(v) \cup \{v\}$. Whenever it is clear from the context what T is, we drop the subscript and write $N(v) = N_T(v)$ and $N[v] = N_T[v]$. For $X \subseteq V(T)$, we write $N(X) = (\cup_{x \in X} N(x)) \setminus X$ and $N[X] = N(X) \cup X$. We say that a set $K \subseteq V(T)$ is a *(strong) clique* if the vertices in K are pairwise (strongly) adjacent. We say that a set $S \subseteq V(T)$ is a *(strong) stable set* if the vertices in S are pairwise (strongly) antiadjacent. The *stability number* $\alpha(T)$ of a trigraph T is the size of a largest stable set in T.

We say that a trigraph T' is a *thickening of* T if for every $v \in V(T)$ there is a nonempty subset $X_v \subseteq V(T')$, all pairwise disjoint and with union $V(T')$, satisfying the following:

(i) for each $v \in V(T)$, X_v is a strong clique of T';
(ii) if $u, v \in V(T)$ are strongly adjacent in T, then X_u is strongly complete to X_v in T';
(iii) if $u, v \in V(T)$ are strongly antiadjacent in T, then X_u is strongly anticomplete to X_v in T';
(iv) if $u, v \in V(T)$ are semiadjacent in T, then X_u is neither strongly complete nor strongly anticomplete to X_v in T'.

When $F(T') = \emptyset$ then we call T' regarded as a graph a *graphic thickening of* T. Observe that if T is a trigraph and G is a graphic thickening of T, then $\alpha(G) = \alpha(T)$.

For $X \subseteq V(T)$, we define the trigraph $T|X$ *induced on* X as follows. The vertex set of $T|X$ is X, and the adjacency function of $T|X$ is the restriction of θ_T to X^2. We call $T|X$ an *induced subtrigraph* of T. We define $T \setminus X = T|(V(T) \setminus X)$. We say that a graph G is a *realization of* T if $V(G) = V(T)$ and for distinct $u, v \in V(T)$, u and v are adjacent in G if u and v are strongly adjacent in T, u and v are nonadjacent in G if u and v are strongly antiadjacent in T, and u and v are either adjacent or nonadjacent in G if u and v are semiadjacent in T. We say that T contains a graph H as a *weakly induced subgraph* if there exists a realization of T that contains H as an induced subgraph. We mention the following easy lemma:

(2.1) (*(2.1) in [3]*). *Let T be a trigraph and let H be a graph. If T contains H as a weakly induced subgraph, then every graphic thickening of T contains H as an induced subgraph.*

A stable set S is called a *triad* if $|S| = 3$. T is said to be *claw-free* if T does not contain the claw as a weakly induced subgraph. A trigraph T is said to be $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a weakly induced subgraph. We state the following trivial result without proof.

(2.2). *Let T be a claw-free trigraph. Then no $v \in V(T)$ is complete to a triad in T.*

Let $p_1, p_2, \ldots, p_k \in V(T)$ be distinct vertices. We say that $T|\{p_1, p_2, \ldots, p_k\}$ of T is a *weakly induced path* (*from p_1 to p_k*) in T if, for $i, j \in [k]$, $i < j$, p_i and p_j are adjacent if $j = i + 1$ and antiadjacent otherwise. Let $\{c_1, c_2, \ldots, c_k\} \subseteq V(T)$. We say that $T|\{c_1, c_2, \ldots, c_k\}$ is a *weakly induced cycle* (*of length k*) in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i - j| = 1 \pmod k$, and antiadjacent otherwise. We say that $T|\{c_1, c_2, \ldots, c_k\}$ is a *semihole* (*of length k*) in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i - j| = 1 \pmod k$, and strongly antiadjacent otherwise. A vertex v in a trigraph T is *simplicial* if $N(v)$ is a strong clique. Notice that our definition of a simplicial vertex differs slightly from the definition used in [5], because we allow v to be incident with a semiedge.

Finally, we say that a set $X \subseteq V(T)$ is a *homogeneous set in T* if $|X| \geq 2$ and $\theta_T(x, v) = \theta_T(x', v)$ for all $x, x' \in X$ and all $v \in V(T) \setminus X$. For two vertices $x, y \in V(T)$, we say that x is a *clone of* y if $\{x, y\}$ is a homogeneous set in T. In that case we say that x and y are *clones*.

2.2. Classes of trigraphs

Let us define some classes of trigraphs:

- **Line trigraphs**. Let H be a graph, and let T be a trigraph with $V(T) = E(H)$. We say that T is a *line trigraph* of H if for all distinct $e, f \in E(H)$:
  - if e, f have a common end in H then they are adjacent in T, and if they have a common end of degree at least three in H, then they are strongly adjacent in T;
  - if e, f have no common end in H then they are strongly antiadjacent in T.
- **Long circular interval trigraphs**. Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval [0, 1], such that no two of $F_1, \ldots, F_k$ share an end-point, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let T be a trigraph with vertex set V in which, for distinct $u, v \in V$,
  - if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent
  - if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

  Such a trigraph T is called a *long circular interval trigraph*.

2.3. A structure theorem for claw-free trigraphs

Let T be a trigraph such that $V(T) = A \cup B \cup C$ and A, B, C are strong cliques. Then (T, A, B, C) is called a *three-cliqued trigraph*. Let (T, A, B, C) be a three-cliqued claw-free trigraph, and let $z \in A$ be such that z is strongly anticomplete to $B \cup C$. Let V_1, V_2, V_3 be three disjoint sets of new vertices, and let T' be the trigraph obtained by adding V_1, V_2, V_3 to T with the following adjacencies:

(i) V_1 and $V_2 \cup V_3$ are strong cliques;
(ii) V_1 is strongly complete to $B \cup C$ and strongly anticomplete to A;
(iii) V_2 is strongly complete to $A \cup C$ and strongly anticomplete to B;
(iv) V_3 is strongly complete to $A \cup B$ and strongly anticomplete to C.

The adjacency between V_1 and $V_2 \cup V_3$ is arbitrary. It follows that T' is claw-free, and z is a simplicial vertex of it. In this case we say that $(T', \{z\})$ is a *hex-expansion* of (T, A, B, C). (See Fig. 6 for an illustration.)

A *multigraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (*i.e.,* a subset of $V(H) \times E(H)$) such that every $F \in E(H)$ is incident with two members of $V(H)$ which are called the *endpoints* of F. For $F \in E(H)$, $\bar{F} = \{u, v\}$ where u, v are the two endpoints of F.

Let T be a trigraph. A *strip-structure* (H, η) of T consists of a multigraph H with $E(H) \neq \emptyset$ (which we call the *pattern multigraph* for the strip-structure), and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(T)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \bar{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

(a) The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(T)$.
(b) For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \bar{F}$ is a strong clique of T.
(c) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in T, then there exists $h \in \bar{F_1} \cap \bar{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

(There is a fourth condition, but we do not need it here.) Let (H, η) be a strip-structure of a trigraph T, and let $F \in E(H)$, where $\bar{F} = \{h_1, h_2\}$. Let v_1, v_2 be two new vertices. Let $Z = \{v_i \mid i \in [2],\ \eta(F, h_i) \neq \emptyset\}$ and let J be the trigraph obtained from $T|\eta(F)$ by adding the vertices in Z, where $v_i \in Z$ is strongly complete to $\eta(F, h_i)$ and strongly anticomplete to all other vertices of J. Then (J, Z) is called the *strip of* (H, η) *at* F. (In the strip-structures that we are interested in, for every $F \in E(H)$ with $\bar{F} = \{h_1, h_2\}$, at least one of $\eta(F, h_1), \eta(F, h_2)$ will be nonempty and therefore $1 \leq |Z| \leq 2$.)

Next, we list the classes of strips (J, Z) that we need for the structure theorem. We call the corresponding sets of pairs $(J, Z)\mathcal{Z}_1$–$\mathcal{Z}_{15}$. (The unnatural ordering of the types of strips is due to the fact that we keep the same ordering as in [5].) The strips marked with a star will turn out (see (4.5)) to contain a weakly induced cycle of length six, and hence are not $\mathcal{F}$-free. See Figs. 7–21 for illustrations of these strips.

$\mathcal{Z}_1$: **(Linear interval strips)**. Let J be a trigraph with vertex set $\{v_1, \ldots, v_n\}$, such that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. We call J a linear interval trigraph. (Every linear interval trigraph is also a long circular interval trigraph.) Also, let $n \geq 2$ and let v_1, v_n be strongly antiadjacent, and let there be no vertex adjacent to both v_1, v_n, and no vertex semiadjacent to either v_1 or v_n. Let $Z = \{v_1, v_n\}$.

$\mathcal{Z}_2$: **(Near antiprismatic strips)**. Let $n \geq 2$. Construct a trigraph J' as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$.

Let the trigraph just constructed be J' and let $J = J' \setminus X$. Let a_0 be strongly antiadjacent to b_0, and let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_3$: **(Line-trigraph strips)**. Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let J be obtained from a line trigraph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of J) either semiadjacent or strongly antiadjacent to each other in J. Let $Z = \{h_1h_2, h_4h_5\}$.

$\mathcal{Z}_4$: **(Sporadic family of trigraphs of bounded size #1)**. Let J be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}$, $\{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; b_2, c_1 are strongly adjacent; b_2, c_2 are semiadjacent; b_3, c_1 are semiadjacent; and all other pairs are strongly antiadjacent. Let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_5$: **(Sporadic family of trigraphs of bounded size #2** *). Let J' be the trigraph with vertex set $\{v_1, \ldots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6-v_1 is a hole in J' of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. No other pairs are adjacent, and all adjacent pairs are strongly adjacent except possibly v_9, v_{10}. (Thus the pair v_9v_{10} is either strongly adjacent or semiadjacent.) Let $J = J' \setminus X$, where $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$, and let $Z = \{v_7, v_8\} \setminus X$.

$\mathcal{Z}_6$: **(Long circular interval strips)**. Let J be a long circular interval trigraph, and let $\Sigma, F_1, \ldots, F_k$ be as in the corresponding definition. Let $z \in V(J)$ belong to at most one of $F_1, \ldots, F_k$, and not be an endpoint of any of $F_1, \ldots, F_k$. Then z is a simplicial vertex of J; let $Z = \{z\}$.

$\mathcal{Z}_7$: (**Modifications of $L(K_6)$**). Let H be a graph with seven vertices $h_1, \ldots, h_7$, in which h_7 is adjacent to h_6 and to no other vertex, h_6 is adjacent to at least three of $h_1, \ldots, h_5$, and there is a cycle with vertices h_1-h_2-$\cdots$-h_5-h_1 in order. Let J' be the graph obtained from the line graph of H by adding one new vertex, adjacent precisely to those members of $E(H)$ that are not incident with h_6 in H. Then J' is a claw-free graph. Let J be either J' (regarded as a trigraph), or (in the case when h_4, h_5 both have degree two in H), the trigraph obtained from J' by making the vertices $h_3h_4, h_1h_5 \in V(J')$ semiadjacent. Let e be the edge h_6h_7 of H, and let $Z = \{e\}$.

$\mathcal{Z}_8$: **(Augmented near antiprismatic strips)**. Let $n \geq 2$. Construct a trigraph J as follows. Its vertex set is the disjoint union of four sets A, B, C and $\{d_1, \ldots, d_5\}$, where $|A| = |B| = |C| = n$, say $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_n\}$. Let $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$. Adjacency is as follows: A, B, C are strong cliques; for $1 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j if and only if $i \neq j$, and c_i is strongly adjacent to b_j if and only if $i \neq j$. Moreover,

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$;
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$;
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$;
- no two of $A \setminus X, B \setminus X, C \setminus X$ are strongly complete to each other.

Also, d_1 is strongly complete to $A \cup B \cup C$; d_2 is strongly complete to $A \cup B$, and either semiadjacent or strongly adjacent to d_1; d_3 is strongly complete to $A \cup \{d_2\}$; d_4 is strongly complete to $B \cup \{d_2, d_3\}$; d_5 is strongly adjacent to d_3, d_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be J'. Let $J = J' \setminus X$ and $Z = \{d_5\}$.

$\mathcal{Z}_9$: **(Special type of antiprismatic strips)**. Let J have a vertex set partitioned into five sets $\{z\}, A, B, C, D$, with $|A| = |B| = n \geq 1$, say $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, such that

- $\{z\} \cup D$ is a strong clique and z is strongly antiadjacent to $A \cup B \cup C$,
- $A \cup C$ and $B \cup C$ are strong cliques,
- for $1 \leq i \leq n$, a_i, b_i are antiadjacent, and every vertex in D is strongly adjacent to exactly one of a_i, b_i and strongly antiadjacent to the other, and
- for $1 \leq i < j \leq n$, $\{a_i, b_i\}$ is strongly complete to $\{a_j, b_j\}$.

(The adjacency between C and D is arbitrary.) Let $Z = \{z\}$.

$\mathcal{Z}_{10}$: **(Sporadic family of trigraphs of bounded size #3)**. Let J' be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2, d\}$ and adjacency as follows: $A = \{a_0, a_1, a_2, d\}$, $B = \{b_0, b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; a_2 is strongly adjacent to b_0 and semiadjacent to b_1; b_2, c_2 are semiadjacent; b_2, c_1 are strongly adjacent; b_3, c_1 are either semiadjacent or strongly adjacent; b_0, d are either semiadjacent or strongly adjacent; and all other pairs are strongly antiadjacent. Then (J', A, B, C) is a three-cliqued trigraph (not claw-free) and a_0 is a simplicial vertex of J'. Let $X \subseteq \{a_2, b_2, b_3, d\}$ such that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$, let $Z = \{a_0\}$, and let (J, Z) be a hex-expansion of $(J' \setminus X, A \setminus X, B \setminus X, C)$.

$\mathcal{Z}_{11}$: **(Hex-expansions of near-antiprismatic trigraphs)**. Let $n \geq 2$. Construct a trigraph J' as follows. Its vertex set is the disjoint union of four sets $\{z\}, A, B, C$, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}$, $B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. z is strongly complete to A and strongly anticomplete to $B \cup C$. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$.

Let the trigraph just constructed be J'. Let $Z = \{z\}$ and let (J, Z) be a hex-expansion of $(J' \setminus X, (A \setminus X) \cup Z, B \setminus X, C \setminus X)$.

$\mathcal{Z}_{12}$: (**Hex-expansions of sporadic exception #2** *). Let J' be the trigraph with vertex set $\{v_1, \ldots, v_9\}$, and adjacency as follows: the sets $A = \{v_3, v_4, v_5, v_6, v_9\}$, $B = \{v_1, v_2\}$ and $C = \{v_7, v_8\}$, are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

- v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$;
- v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$;
- if $v_4, v_5 \in X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Let J'' be the trigraph obtained from J' be adding a new vertex z that is strongly complete to A. Let $Z = \{z\}$. Then $(J'' \setminus X, (A \cup Z) \setminus X, B, C)$ is a three-cliqued trigraph. Let (J, Z) be a hex-expansion of $(J'' \setminus X, (A \cup Z) \setminus X, B, C)$.

$\mathcal{Z}_{13}$: **(Hex-expansions of long circular interval trigraphs)**. Let J' be a long circular interval trigraph such that every vertex of J' is in a triad, and let Σ be a circle with $V(J') \subseteq \Sigma$, and $F_1, \ldots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(J)$ with $|X| \leq 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(J)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(J') \subseteq L_1 \cup L_2 \cup L_3$. Then $(J', V(J') \cap L_1, V(J') \cap L_2, V(J') \cap L_3)$ is a three-cliqued claw-free trigraph. Let $z \in L_1$ belong to the interior of F_1. Thus, z is a simplicial vertex of J'. Let $Z = \{z\}$ and let (J, Z) be a hex-expansion of $(J', V(J') \cap L_1, V(J') \cap L_2, V(J') \cap L_3)$.

$\mathcal{Z}_{14}$: (**Hex-expansions of line trigraphs** *). Let v_0, v_1, v_2, v_3 be distinct vertices of a graph H, such that: v_1 is the only neighbor of v_0 in H; every vertex of H different from v_0, v_1, v_2, v_3 is adjacent to both v_2, v_3, and at most one of them is nonadjacent to v_1; v_1, v_2, v_3 are pairwise nonadjacent, and each has degree at least three. For $i = 1, 2, 3$, let A_i be the set of edges of H incident with v_i, and let z be the edge v_0v_1. Let J' be a line trigraph of H; thus (J', A_1, A_2, A_3) is a three-cliqued claw-free trigraph, and z is a simplicial vertex of J'. Let $Z = \{z\}$, and let (J, Z) be a hex-expansion of (J', A_1, A_2, A_3).

$\mathcal{Z}_{15}$: **(Hex-expansions of sporadic exception #1)**. Let J' be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \le i < j \le 6$ with $j - i \le 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; $\{v_1, v_6, v_7\}$ is a strong clique, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \dots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}$, $B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(J' \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued trigraph and all its vertices are in triads. Let $Z = \{v_8\}$ and let (J, Z) be a hex-expansion of $(J' \setminus X, A \setminus X, B \setminus X, C)$.

Notice that only the elements of $\mathcal{Z}_1, \dots, \mathcal{Z}_5$ have $|Z| = 2$. Informally speaking, this means that such strips are the only strips that can (but not necessarily do) attach to the rest of the trigraph on two sides (through a so-called '2-join', see Section 5). The other types of strips, $\mathcal{Z}_6, \dots, \mathcal{Z}_{15}$, have $|Z| = 1$ and attach to the rest of the trigraph on one side (through a so-called '1-join'). Also notice that the strips in $\mathcal{Z}_2, \dots, \mathcal{Z}_5$ are three-cliqued.

Let $\mathcal{Z}_0 = \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_{15}$. We say that a claw-free trigraph T is *basic* if T is a trigraph from the icosahedron, an antiprismatic trigraph, a long circular interval trigraph, or a trigraph that is a union of three strong cliques (since their definitions are long and irrelevant for the current paper, we refer to [3] for the definitions), and T is *nonbasic* otherwise (we will describe the structure of such nonbasic trigraph completely). Analogously, a claw-free graph G is *basic* if G is a graphic thickening of a basic claw-free trigraph T and G is *nonbasic* otherwise.

Let $F \in E(H)$ and let (J, Z) be the strip of (H, η) at F. We say that (J, Z) is a *spot* if $\eta(F) = \eta(F, u) = \eta(F, v)$ and $|\eta(F)| = 1$. Let J' be a thickening of J and, for $v \in V(J)$, let X_v be the strong clique in J' that corresponds to v. Let $Z' = \bigcup_{z \in Z} X_z$. If $|X_z| = 1$ for each $z \in Z$, then we say that (J', Z') is a *thickening of* (J, Z).

We say that a strip-structure (H, η) is *proper* if all of the following hold:

(1) $|E(H)| \ge 2$; (a strip-structure that satisfies only this condition is called *nontrivial* in [5])
(2) for each strip (J, Z), either
  (a) (J, Z) is a spot, or
  (b) (J, Z) is a thickening of a member of $\mathcal{Z}_0$;
(3) for every $F \in E(H)$, if the strip of (H, η) at F is a thickening of a member of $\mathcal{Z}_6 \cup \mathcal{Z}_7 \cup \dots \cup \mathcal{Z}_{15}$, then, at least one of the vertices in $\bar{F}$ has degree 1.

We note that in the definition of a strip-structure (H, η) given in [5], the multigraph H is actually a hypergraph. In this hypergraph, however, every hyperedge has cardinality either one or two. We may replace every hyperedge F of cardinality one by a new vertex, z say, and a new edge F' with $\{u, z\}$, where u is the unique vertex in $\bar{F}$, and setting $\eta(F') = \eta(F)$, $\eta(F', u) = \eta(F, u)$, and $\eta(F, z) = \emptyset$. Thus, we may regard this hypergraph as a multigraph. With this observation in mind, the following theorem is an easy corollary of the main result of [5].

(2.3) *([5]). Every connected nonbasic claw-free graph is a graphic thickening of a claw-free trigraph that admits a proper strip-structure.*

2.4. Resolved graphs and trigraphs; finding dominant cliques

We say that an $\mathcal{F}$-free claw-free trigraph T is *resolved* if every $\mathcal{F}$-free thickening of T is resolved. We state a number of useful lemmas for concluding that a trigraph is resolved. Let T be a trigraph. For a vertex $x \in V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* x if x has a neighbor in S. For a strong clique $K \subseteq V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* K if S covers every vertex in K. We say that a strong clique $K \subseteq V(T)$ is a *dominant clique* if T contains no stable set $S \subseteq V(T) \setminus K$ such that S covers K. It is easy to see that this definition of a dominant clique, when applied to a graph, coincides with our earlier definition of a dominant clique for a graph.

(2.6) *((2.3) in [3]). Let T be a trigraph and suppose that K is a dominant clique in T. Then, T is resolved.*

(2.7) *((2.4) in [3]). Let T be a trigraph, let A and B be nonempty disjoint strong cliques in T and suppose that B is strongly anticomplete to $V(T) \setminus (A \cup B)$. Then, T is resolved.*

(2.8) *((2.5) in [3]). Let T be a trigraph and let $v \in V(T)$ be a simplicial vertex in T. Then, T is resolved.*

(2.9) *((2.6) in [3]). Let T be a trigraph with no triad. Then, T is resolved.*

Let T be a trigraph, and suppose that X_1 and X_2 are disjoint nonempty strong cliques. We say that (X_1, X_2) is a *homogeneous pair of cliques* in T if, for $i = 1, 2$, every vertex in $V(T) \setminus (X_1 \cup X_2)$ is either strongly complete or strongly anticomplete to X_i. For notational convenience, for a weakly induced path $P = p_1\text{-}p_2\text{-}\cdots\text{-}p_{k-1}\text{-}p_k$, define the *interior* P^* *of* P by $P^* = p_2\text{-}p_3\text{-}\cdots\text{-}p_{k-2}\text{-}p_{k-1}$.

(2.10) *((2.7) in [3]). Let T be an $\mathcal{F}$-free claw-free trigraph. Let (K_1, K_2) be a homogeneous pair of cliques in T such that K_1 is not strongly complete and not strongly anticomplete to K_2. For $\{i, j\} = \{1, 2\}$, let $N_i = N(K_i) \setminus N[K_j]$ and $M = V(T) \setminus (N[K_1] \cup N[K_2])$. If there exists a weakly induced path P between antiadjacent $v_1 \in N_1$ and $v_2 \in N_2$ such that $V(P^*) \subseteq M$ and $|V(P)| \ge 3$, then T is resolved.*

(2.11) *((2.8) in [3]). Let T be an $\mathcal{F}$-free claw-free trigraph and suppose that T contains a weakly induced cycle c_1-c_2-$\cdots$-c_k-c_1 with $k \geq 5$ and such that $c_1c_2 \in F(T)$. Then, T is resolved.*

(2.12) *((2.9) in [3]). Let T be a trigraph and suppose that $v, w \in V(T)$ are strongly adjacent clones. If $T \setminus v$ is resolved, then T is resolved.*

3. The proof of Theorem 1.1 assuming Theorem 1.2

The remainder of this paper is devoted to proving Theorem 1.2. This suffices because Theorem 1.2 implies our main theorem, Theorem 1.1:

Proof of Theorem 1.1 (*Assuming* Theorem 1.2). Theorem 1.3 in [3] proves that (i) implies (ii). Theorem 1.4 in [3] proves that (iii) implies (i). Thus, it suffices to prove that (ii) implies (iii). So let G be an $\mathcal{F}$-free claw-free graph and let G' be any connected induced subgraph of G. It follows from (2.1) and (2.3) that G' is a graphic thickening of an $\mathcal{F}$-free claw-free trigraph T. G' is resolved by Theorem 1.4 in [3] if T is basic, and by Theorem 1.2 if T is nonbasic. This proves that every connected induced subgraph of G is resolved, and therefore that G is perfectly resolved. This proves that (ii) implies (iii), thereby completing the proof of Theorem 1.1. □

4. A structure theorem for the pattern multigraph for the strip-structure of $\mathcal{F}$-free claw-free trigraphs

Let G be a nonbasic claw-free graph. We say that (T, H, η) is a *representation* of G if G is a graphic thickening of T, and (H, η) is a proper strip-structure for T. We say that a representation is *optimal* for G if T is not a thickening of any other claw-free trigraph and, subject to that, H has a maximum number of edges.

Observation 4.1. *Let (T, H, η) be an optimal representation of some claw-free graph G. Then, by the fact that T is not the thickening of some other trigraph, T has no strongly adjacent clones. In particular, (H, η) has no parallel spots. Moreover, every strip (J, Z) is connected (this follows from the definition of $\mathcal{Z}_0$ and the maximality of the number of edges).*

Let H be a multigraph. We say that a vertex $x \in V(H)$ is a *cut-vertex of* H if $H \setminus \{x\}$ is disconnected. A multigraph H is 2-*connected* if H has no cut-vertex. A maximal submultigraph of H that has no cut-vertex is called a *block* of H, and the collection $(B_1, \ldots, B_q)$ of blocks of H is called the *block-decomposition of* H. It is well-known that the block-decomposition of a multigraph exists and is unique (see *e.g.*, [9]). Observe that a multigraph H is 2-connected if and only if H has at most one block. For a cycle C in H and $F \in E(C)$, let $C \setminus F$ denote the graph obtained from C by deleting F.

Let G be a graph and let $x \in V(G)$. Construct G' by adding a vertex x' such that $N(x) = N(x')$. Then, we say x and x' are *nonadjacent* clones in G' and we say that G' is constructed from G by *nonadjacent cloning of* x. Let $t \geq 1$. Let K_t be a complete graph on t vertices. Let $K_{2,t}$ denote a complete bipartite graph whose vertex set is the union of disjoint stable sets X, Y with $|X| = 2$ and $|Y| = t$. Let $K^+_{2,t}$ denote the graph constructed from $K_{2,t}$ by adding an edge between the two vertices in X, where X is as in the definition of $K_{2,t}$.

We define the following two classes of graphs:

$\mathcal{B}_1$: Let us first define the class $\mathcal{B}^*_1$. Let $k \in \{5, 7\}$ and let G be a graph with vertex set $\{c_1, c_2, \ldots, c_k\}$ such that c_1-c_2-$\cdots$ c_k-c_1 is a cycle. If $k = 5$, then each other pair not specified so far is either adjacent or nonadjacent. If $k = 7$, then all pairs that are not in the cycle are nonadjacent except possibly a subset of the pairs $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_4, c_7\}$. Then, $G \in \mathcal{B}^*_1$.
Now let every graph in $\mathcal{B}^*_1$ be in $\mathcal{B}_1$. For every $G' \in \mathcal{B}_1$, let the graph G'' constructed from G' by nonadjacent cloning of a vertex of degree 2 be in $\mathcal{B}_1$.

$\mathcal{B}_2$: Let $\mathcal{B}_2 = \{K_2, K_3, K_4, K_{2,t}, K^+_{2,t} \mid t \geq 2\}$.

For a multigraph H, let $U(H)$ be the graph constructed from H by removing all but one in each class of parallel edges and regarding the resulting multigraph as a graph. For $i \in [2]$, we say that a multigraph H is *of the $\mathcal{B}_i$ type* if $U(H)$ is isomorphic to a graph in $\mathcal{B}_i$. It turns out that if (T, H, η) is an optimal representation of an $\mathcal{F}$-free nonbasic claw-free graph, then the structure of H is relatively simple. In particular, the goal of this section is to prove the following:

(4.2). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph. Then, G has an optimal representation and, for every optimal representation (T, H, η), all of the following hold:*

(i) *every block of H is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type;*
(ii) *at most one block of H is of the $\mathcal{B}_1$ type;*
(iii) *for every cycle C in H with $|E(C)| \geq 4$, all strips of (H, η) at $F \in E(C)$ are spots.*

Fig. 3 illustrates the structure of H. The block decomposition of the multigraph H shown in the figure has one block of the $\mathcal{B}_1$ type. The other blocks are of the $\mathcal{B}_2$ type.

4.1. Properties of optimal representations of $\mathcal{F}$-free nonbasic claw-free graphs

Before we can prove (4.2), we need to prove some lemmas. We use the results in this subsection later on as well.

(4.3). *Let G be an $\mathcal{F}$-free claw-free graph and let (T, H, η) be an optimal representation of G. Then, for each strip (J, Z), either*

(a) *(J, Z) is a spot, or*
(b) *(J, Z) is isomorphic to a member of $\mathcal{Z}_0$.*

Proof. Suppose that, for some $F \in E(H)$, (J, Z) is not a spot and (J, Z) is not isomorphic to a member of $\mathcal{Z}_0$. Then, (J, Z) is a thickening of some member (J', Z') of $\mathcal{Z}_0$. Now, construct (T', H, η') by replacing (J, Z) by (J', Z'), and updating the corresponding sets for η. Then, G is a graphic thickening of T' and T is a thickening of T', contrary to the fact that (T, H, η) is an optimal representation for G. This proves (4.3). $\square$

The following lemma states that T and every strip of the strip-structure is $\mathcal{F}$-free (recall that a trigraph T is $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a weakly induced subgraph).

(4.4). *Let (T, H, η) be a representation of some $\mathcal{F}$-free claw-free graph G. Then T is $\mathcal{F}$-free and, for all $F \in E(H)$, the strip of (H, η) at F is $\mathcal{F}$-free.*

Proof. It follows from (2.1) that if T contains a graph $H \in \mathcal{F}$ as a weakly induced subgraph, then G contains H as an induced subgraph, a contradiction. Therefore, T is $\mathcal{F}$-free. Next, let $F \in E(H)$ and consider the strip (J, Z) of (H, η) at F and suppose that for some $X \subseteq V(J)$, $J|X$ contains a graph $H \in \mathcal{F}$ as a weakly induced subgraph. We may choose X minimal with this property. Because none of the graphs in $\mathcal{F}$ has a simplicial vertex, it follows that $X \cap Z = \emptyset$. Therefore, $J|X$ is an induced subtrigraph of T that contains H as a weakly induced subgraph, contrary to the fact that T is $\mathcal{F}$-free. This proves (4.4). $\square$

(4.4) implies that three classes of strips do not occur in the strip-structure of $\mathcal{F}$-free claw-free trigraphs, more precisely:

(4.5). *Let (T, H, η) be a representation of some $\mathcal{F}$-free claw-free graph. Let $F \in E(H)$. Then, the strip of (H, η) at F is not isomorphic to a member of $\mathcal{Z}_5 \cup \mathcal{Z}_{12} \cup \mathcal{Z}_{14}$.*

Proof. Suppose that the strip of (H, η) at F is isomorphic to a member $(J, Z) \in \mathcal{Z}_5$. For $i \in [6]$, let v_i be as in the definition of $\mathcal{Z}_5$. Then, v_1-v_2-$\cdots$-v_6-v_1 is a weakly induced cycle of length six in J, contrary to (4.4). Next, suppose that $(J, Z) \in \mathcal{Z}_{12}$. Let $v_1, v_2, \ldots, v_9, X$ be as in the definition of $\mathcal{Z}_{12}$. Let $j \in \{3, 4\}$ be largest such that v_2 is adjacent to v_j and let $k \in \{5, 6\}$ be smallest such that v_7 is adjacent to v_k. Such j, k exist by the fact that v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$ and v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$. But now v_1-v_2-v_j-v_k-v_7-v_8-v_1 is a weakly induced cycle of length six in J, contrary to (4.4). Finally, suppose that the strip of (H, η) at F is isomorphic to a member $(J, Z) \in \mathcal{Z}_{14}$. Let $H', T', v_0, v_1, v_2, v_3$ be as in the definition of $\mathcal{Z}_{14}$. Let $N = V(H') \setminus \{v_0, v_1, v_2, v_3\}$. Because $\deg(v_i) \geq 3$, for $i = 1, 2, 3$, there exist p_1, p_2, p_3 such that p_1, p_2 are complete to $\{v_1, v_2, v_3\}$ and p_3 is complete to $\{v_2, v_3\}$. Now v_1-p_1-v_2-p_3-v_3-p_2-v_1 is a cycle of length six in H'. Hence, T' has a weakly induced cycle of length six. Thus, J has a weakly induced cycle of length six, contrary to (4.4). $\square$

Let (T, H, η) be an optimal representation of some nonbasic claw-free graph. Let $F \in E(H)$ and let $\{u, v\} = \bar{F}$. Let $\ell(F)$ denote the set of integers k such that there exists a k-vertex weakly induced path from a vertex in $\eta(F, u)$ to a vertex to $\eta(F, v)$ whose interior vertices lie in $\eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. Notice that for $F \in E(H)$ with $\{u, v\} = \bar{F}$, $\ell(F) = \emptyset$ if and only if one of $\eta(F, u)$ or $\eta(F, v)$ is empty (the strip of (H, η) at such an F is a thickening of a member of $\mathcal{Z}_5 \cup \mathcal{Z}_6 \cup \cdots \cup \mathcal{Z}_{15}$). For a set of edges $S \subseteq E(H)$, we define

$$\ell(S) = \left\{ \sum_{F \in S} x_F \mid x_F \in \ell(F), F \in S \right\}.$$

To clarify, $\ell(S)$ is the set of numbers that can be obtained by choosing for each $F \in S$ a number $x_F \in \ell(F)$ and taking the sum of these numbers $\{x_F\}_{F \in S}$. We have the following property:

(4.6). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph. Let $F \in E(H)$ and $\{u, v\} = \bar{F}$. The following statements are equivalent:*

(i) *$1 \in \ell(F)$,*
(ii) *$\eta(F, u) \cap \eta(F, v) \neq \emptyset$,*
(iii) *$\ell(F) = \{1\}$,*
(iv) *the strip of (H, η) corresponding to F is a spot.*

Moreover, if $\ell(F) \neq \emptyset$, then the strip of (H, η) at F is either a spot, or isomorphic to a member of $\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_4$.

Proof. Clearly, (i) and (ii) are equivalent. Moreover, it is clear that (iv) implies (iii) and (iii) implies (i). Therefore, it suffices to prove that (ii) implies (iv). So suppose that $\eta(F, u) \cap \eta(F, v) \neq \emptyset$. If the strip (J, Z) of (H, η) at F is a spot, we are done. So we may assume that (J, Z) is not a spot and hence, by (4.3), (J, Z) is isomorphic to a member of $\mathcal{Z}_0$. Since $\eta(F, u) \cap \eta(F, v) \neq \emptyset$ and, in particular, $\eta(F, u)$ and $\eta(F, v)$ are both nonempty, it follows that (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_5$. Let $\{z_1, z_2\} = Z$ and let $x \in \eta(F, u) \cap \eta(F, v) \subseteq V(J)$. It follows that x is a common neighbor of z_1 and z_2, but it is easy to check from the definitions of $\mathcal{Z}_1, \ldots, \mathcal{Z}_5$ that J does not have such a vertex.

For the second statement, suppose that $\ell(F) \neq \emptyset$. If the strip of (H, η) at F is a spot, then we are done. So we may assume by the definition of a proper strip-structure that the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_0$. Let $\{u, v\} = \bar{F}$. The fact that $\ell(F) \neq \emptyset$ implies that $\eta(F, u)$ and $\eta(F, v)$ are both nonempty. Therefore, the strip (J, Z) of (H, η) at F satisfies $|Z| = 2$, and hence (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_5$. Moreover, (J, Z) is not isomorphic to a member of $\mathcal{Z}_5$ because of (4.5). This proves (4.6). $\square$

A *cycle C in H* is a subgraph of H on vertex set $\{c_1, c_2, \ldots, c_k\}$, with $k \geq 2$, and edge set $\{F_1, F_2, \ldots, F_k\}$ such that $\bar{F}_i = \{c_i, c_{i+1}\}$ (with subscript modulo k). Notice that, by property (3) of the definition of a proper strip-structure, it follows that, for every cycle C in H, $\ell(F) \neq \emptyset$ for all $F \in E(C)$ and, thus, $\ell(E(C)) \neq \emptyset$. The following lemma deals with the possible values of $\ell(E(C))$ for cycles C in H.

(4.7). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph. Let C be a cycle in H. Then, $z \in \{3, 4, 5, 7\}$ and $z \geq |E(C)|$ for all $z \in \ell(E(C))$.*

Proof. Suppose for a contradiction that there exists $z \in \ell(E(C)) \setminus \{3, 4, 5, 7\}$. Assume first that $z = 2$. Then it follows that $|E(C)| = 2$ and hence that the strips corresponding to the edges of C are spots. Let $F \in E(C)$. Clearly, T is a thickening of $T \setminus \eta(F)$, which contradicts the fact that (T, H, η) is an optimal representation. Hence, $z = 6$ or $z \geq 8$. Now, write $C = c_1\text{-}c_2\text{-}\cdots\text{-}c_k\text{-}c_1$ with $k = |E(C)|$ and such that, for all $i \in [k]$, there exists $F_i \in E(C)$ with $\bar{F}_i = \{c_i, c_{i+1}\}$ (subscripts modulo k). For $i \in [k]$, let $x_i \in \ell(F_i)$ be such that $z = \sum_{i\in[k]} x_i$ and let P_i be a weakly induced path from a vertex in $\eta(F_i, c_i)$ to a vertex in $\eta(F_i, c_{i+1})$ with $|V(P_i)| = x_i$. Now, $P_1\text{-}P_2\text{-}\cdots\text{-}P_k\text{-}P_1$ is a weakly induced cycle of length z, a contradiction. This proves (4.7). $\square$

We would like to stress here that (4.7) shows that optimal strip-structures do not have parallel spots. We need another lemma. For a trigraph T and a set $X \subseteq V(T)$, we say that $y \in V(T) \setminus X$ *is mixed on* X if y is not strongly complete or strongly anticomplete to X. We say that a set $Y \subseteq V(T) \setminus X$ *is mixed on* X if some vertex in Y is mixed on X.

(4.8). *Let T be a claw-free trigraph, and let $A, B, C \subseteq V(T)$ be disjoint nonempty sets in T such that A is strongly anticomplete to B, and C is a clique. Then, either at most one of A, B is mixed on C, or there exists a weakly induced 4-vertex path P with one endpoint in A and the other in B, and $V(P^*) \subseteq C$.*

Proof. Clearly, if $|C| = 1$, then it follows immediately from the fact that no vertex is incident with two semiedges that at most one of A, B is mixed on C. So we may assume that $|C| \geq 2$. We may assume that there exist $a \in A$ and $b \in B$ that are mixed on C. If a is complete to C, then let $X \subseteq C$ be the set of strong neighbors of a in C and let $Y \subseteq C$ be the set of antineighbors of a in C. If a is not complete to C, then let $X \subseteq C$ be the set of neighbors of a in C and let $Y \subseteq C$ be the set of strong antineighbors of a in C. Observe that $C = X \cup Y$ and, because $|C| \geq 2$ and a is mixed on C, X and Y are nonempty. If b has both an antineighbor $x \in X$ and a neighbor in $y \in Y$, then $P = a\text{-}x\text{-}y\text{-}b$ is a weakly induced 4-vertex path with one endpoint in A and the other in B, and $V(P^*) \subseteq C$. Thus, we may assume that b is either strongly complete to X or strongly anticomplete to Y. Next, if b has both a neighbor $x' \in X$ and an antineighbor $y' \in Y$, then x' is complete to the triad $\{a, y', b\}$, contrary to (2.2). It follows that if b is strongly complete to X, then b is strongly complete to Y and, thus, b is not mixed on C. So we may assume that b is strongly anticomplete to Y. But now, it follows that b is strongly anticomplete to X and, thus, b is not mixed on C. It follows that B is not mixed on C, thereby proving (4.8). $\square$

This lemma allows us to rule out strips in which all weakly induced paths have the same length $k \geq 3$. The idea of the proof is that when this happens, the strip has a special structure that allows us to enlarge the strip-structure, therefore showing that the strip-structure we started with was not optimal.

(4.9). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Then, there exists no $F \in E(H)$ such that $\ell(F) = \{k\}$ for some $k \geq 3$.*

Proof. Assume for a contradiction that there exists $F \in E(H)$ such that $\ell(F) = \{k\}$ for some $k \geq 3$. Let (J, Z) be the strip of (H, η) at F. Let $\{u, v\} = \bar{F}$, $A = \eta(F, u)$, $B = \eta(F, v)$, and $C = \eta(F) \setminus (A \cup B)$. It follows from the fact that $1, 2 \notin \ell(F)$ that A and B are disjoint and A is strongly anticomplete to B.

Define the following sets. Let $N_0 = \{z_1\}$ and $N_{k+1} = \{z_2\}$, where z_1 is the unique vertex in Z that is strongly complete to A and z_2 is the unique vertex in Z that is strongly complete to B. Let $N_1 = A$ and $N_k = B$, and let $N_2, \ldots, N_{k-1}$ be such that N_i is strongly anticomplete to N_j if $i < j - 1$, and N_i and N_{i+1} are linked. We may choose $N_2, \ldots, N_{k-1}$ with maximal union and, since there exists a weakly induced path of length k from a vertex in $N_1 = A$ to a vertex in $N_k = B$, $|N_i| \geq 1$ for all $i \in [k]$.

Since $\ell(F) \neq \emptyset$, it follows from (4.6) that (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$. In particular, J is either a linear interval trigraph or a three-cliqued trigraph.

(i) *Let $x \in \eta(F) \setminus (N_1 \cup \cdots \cup N_k)$. Then, there exists $i \in [k-1]$ such that x has a neighbor in N_i and in N_{i+1} and x is anticomplete to N_j with $j \neq i, i+1$.*

Since J is either a linear interval trigraph or a three-cliqued trigraph, it follows that x has a neighbor in $\bigcup_{i=1}^{k} N_i$. Let i be smallest such that x has a neighbor in N_i, say y, and let j be largest such that x has a neighbor in N_j. Clearly, since Z is strongly anticomplete to C, it follows that $1 \leq i \leq j \leq k$. First suppose that $i = j$. Then y has a neighbor $y_1 \in N_{i-1}$

and a neighbor $y_2 \in N_{i+1}$. But now, y is complete to the triad $\{x, y_1, y_2\}$, contrary to (2.2). Thus, $i \neq j$. If $|i-j| = 1$, then the lemma holds. Next, suppose that $|i-j| = 2$. Then, adding x to N_{i+1} contradicts the maximality of $N_1 \cup \cdots \cup N_k$. Thus, $|i-j| \geq 3$. But now, let P_1 be a weakly induced i-vertex path from a vertex in N_1 to a vertex in N_i, and let P_2 be a $(k-j)$-vertex path from a vertex in N_j to a vertex in B. Then, P_1-x-P_2 is a weakly induced path from a vertex in A to a vertex in B that has less that k vertices, a contradiction. This proves (i). □

Next, for $i = 0, 1, \ldots, k$, let $M_{i,i+1} \subseteq \eta(F) \setminus (N_1 \cup \cdots \cup N_k)$ be the set of vertices with a neighbor in both N_i and N_{i+1}. It follows from (i) that $\eta(F) = (\bigcup_{i=1}^{k} N_i) \cup (\bigcup_{i=1}^{k-1} M_{i,i+1})$. Also observe that $M_{0,1} = M_{k,k+1} = \emptyset$.

(ii) *For distinct $i, j \in [k-1]$, $M_{i,i+1}$ is strongly anticomplete to $M_{j,j+1}$.*

Suppose that $x \in M_{i,i+1}$ is adjacent to $y \in M_{j,j+1}$ for distinct $i, j \in [k-1]$. From the symmetry, we may assume that $i < j$. Now, let P_1 be a weakly induced i-vertex path from a vertex in N_1 to a vertex in N_i, and let P_2 be a $(k-j-1)$-vertex path from a vertex in N_{j+1} to a vertex in B. Then, P_1-x-y-P_2 is a weakly induced path from a vertex in A to a vertex in B that has $k' \neq k$ vertices, a contradiction. □

(iii) *For $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique.*

Recall that J is either a linear interval trigraph or a J is three-cliqued. If J is three-cliqued, then it follows from the definitions of the strips that C is a strong clique. So we may assume that J is a linear interval trigraph. Thus, there exists a linear ordering $(\leq, V(J))$ such that for all adjacent $x, y \in V(J)$ and $z \in V(J)$, $x < z \leq y$ implies that z is strongly adjacent to x and y. We may assume that for every $x, y \in V(J)$, either $x > y$ or $x < y$. We prove a stronger statement:

($*$) *For $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique and $v_{i-1} < v_i$ for all adjacent $v_{i-1} \in N_{i-1}$, $v_i \in N_i \cup M_{i-1,i}$.*

We prove ($*$) by induction on i. First consider the case $i = 1$. $N_1 \cup M_{0,1}$ is a strong clique because $N_1 \cup M_{0,1} = A$, and it follows from our assumptions that $v_0 < v_1$ for all $v_0 \in N_0$ and $v_1 \in N_1 \cup M_{0,1}$. So let $i \geq 2$. We first claim that $v_{i-1} < v_i$ for all adjacent $v_{i-1} \in N_{i-1}$ and $v_i \in N_i \cup M_{i-1,i}$. For let $v_{i-1} \in N_{i-1}$ and $v_i \in N_i \cup M_{i-1,i}$ be adjacent. It follows from the definitions of N_{i-1}, N_i, and $M_{i-1,i}$ that v_{i-1} has a neighbor $v_{i-2} \in N_{i-2}$, and v_i is strongly antiadjacent to v_{i-2}. Inductively, $v_{i-2} < v_{i-1}$. Then it follows from the definition of a linear interval trigraph that $v_i > v_{i-1}$, as required.

Now suppose that $N_i \cup M_{i-1,i}$ is not a strong clique. Then there exist antiadjacent $x_1, x_2 \in N_i \cup M_{i-1,i}$. By the definition of N_i and $M_{i-1,i}$, x_1 and x_2 have neighbors $y_1, y_2 \in N_{i-1}$, and y_1, y_2 have neighbors $z_1, z_2 \in N_{i-2}$, where possibly $z_1 = z_2$. Inductively, y_1 and y_2 are strongly adjacent. Since T is claw-free, it follows that both y_1, y_2 are not complete to $\{x_1, x_2\}$. Thus, $y_1 \neq y_2$, y_1 is strongly antiadjacent to x_2 and y_2 is strongly antiadjacent to x_1. It follows from the previous argument that $x_1 > y_1$ and $x_2 > y_2$. From the symmetry between x_1 and x_2, we may assume that $x_1 > x_2$. If $y_1 > x_2$, then the fact that $y_1 > x_2 > y_2$ and y_1 is adjacent to y_2 implies that x_2 is adjacent to y_1, a contradiction. Hence, $y_1 < x_2$. Now, $y_1 < x_2 < x_1$ and the fact that y_1 and x_1 are adjacent imply that x_2 is strongly adjacent to both y_1 and x_1, a contradiction. Thus, N_i is a strong clique. This proves (iii). □

It follows from (iii) that, for $i \in [k-1]$, $N_i \cup M_{i-1,i}$ is a strong clique. From the symmetry, it follows that for $i \in [k] \setminus \{1\}$, $N_i \cup M_{i,i+1}$ is a strong clique. Thus, all sets $M_{i,i+1}$ are strong cliques and each $M_{i,i+1}$ is strongly complete to $N_i \cup N_{i+1}$.

(iv) *If, for some $j \in [k]$, N_j is strongly complete to N_{j+1}, then (T, H, η) is not an optimal representation of G.*

Let $j \in [k]$ be such that N_j is strongly complete to N_{j+1}. Construct a new strip-structure (H', η') for T from (H, η) as follows. First add to H' two new vertices w_1, w_2. Next, replace F by two new edges F_1, F_2 such that $\bar{F}_1 = \{u, w_1\}$, $\bar{F}_2 = \{v, w_1\}$. Let $\eta'(F_1) = \bigcup_{i=1}^{j} (N_i \cup M_{i-1,i})$, $\eta'(F_1, u) = A$, $\eta'(F_1, w_1) = N_j$, $\eta'(F_2) = \bigcup_{i=j+1}^{k} (N_i \cup M_{i,i+1})$, $\eta'(F_2, v) = B$, and $\eta'(F_2, w_1) = N_{j+1}$. If $M_{j,j+1} \neq \emptyset$, it follows from the fact that T is not a thickening of some other claw-free graph that $|M_{j,j+1}| = 1$; now add to H' an edge F_3 with $\bar{F}_3 = \{w_1, w_2\}$ $\eta'(F_3) = \eta'(F_3, w_1) = \eta'(F_3, w_2) = \{z\}$, where z is the unique vertex in $M_{j,j+1}$. Then, the strip of (H', η') at F_1, F_2 is isomorphic to a member of $\mathcal{Z}_1$, and, if $M_{j,j+1} \neq \emptyset$, the strip of (H', η') at F_3 is a spot. Thus, (T, H', η') is representation of G that satisfies $|E(H')| > |E(H)|$ and therefore, (T, H, η) is not an optimal representation, a contradiction. This proves (iv). □

It follows from (4.8) that either at most one of N_1, N_3 is mixed on N_2, or there exists a weakly induced 4-vertex path $P = p_1$-p_2-p_3-p_4 with $p_1 \in A$, $p_2, p_3 \in N_2$, and $p_4 \in N_3$. If such P exists, then clearly, this path may be extended to obtain a $(k+1)$-vertex path from p_1 to a vertex in B, a contradiction. Thus, it follows that at least one of N_1, N_3 is not mixed on N_2. Since N_i and N_{i+1} are linked, it follows that at least one of N_1, N_3 is strongly complete to N_2, and thus the lemma holds by (iv). This proves (4.9). □

The previous lemma deals with strips in which all weakly induced paths have the same length $k \geq 3$. A question is: what happens when all weakly induced paths have length two? The next lemma deals with this case when such a strip is part of a long cycle. The idea of the proof is again that under some circumstances, we may enlarge the strip-structure.

(4.10). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H. If there exists $F \in E(C)$ such that $\ell(F) \in \{\{2\}, \{2, 4\}\}$, then $\ell(E(C \setminus F)) \cap \{3, 5\} = \emptyset$.*

Proof. Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H and let $F \in E(C)$ be such that $\ell(F) \in \{\{2\}, \{2, 4\}\}$. Let $\{u, v\} = \bar{F}$ and let $A' = \eta(F, u)$, $B' = \eta(F, v)$, and $D' = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. We start with the following claim:

(i) *D' is a strong clique.*

Since $\ell(F) \neq \emptyset$, it follows from (4.6) that the strip (J, Z) of (H, η) at F is isomorphic to a member of $\mathcal{Z}_l$ for some $l \in [4]$. If $l \in \{2, 3, 4\}$, then it follows immediately from the definition of the respective strips that D' is a strong clique. So we may assume that $l = 1$. Thus, J is a linear interval trigraph. Since $2 \in \ell(F)$, there exists adjacent $a \in A'$ and $b \in B'$. Now, it follows from the definition of a linear interval trigraph that D' is a strong clique. This proves (i). □

We need to consider the graph G. Recall that G is a graphic thickening of T. For $u \in V(T)$, let X_u be the clique in G that corresponds to u. Let $A = \bigcup\{X_v \mid v \in A'\}$, and define B and D analogously.

(ii) *No vertex in D has nonadjacent neighbors $a \in A$ and $b \in B$.*

If $d \in D$ is adjacent to some nonadjacent $a \in A$ and $b \in B$, then a-d-b is an induced path that implies that $3 \in \ell(F)$, a contradiction. This proves (ii). □

Assume for a contradiction that there exists $m \in \ell(E(C) \setminus \{F\})$ with $m \in \{3, 5\}$. It follows from the definition of a strip-structure that there exists a path p_1-p_2-$\cdots$-p_m in G such that p_1 is complete to B, p_m is complete to A, $V(P^*)$ is anticomplete to $A \cup B$, and $V(P)$ is anticomplete to D. Let $A_0, A_1, A_2, \ldots, A_k \subseteq A$ and $B_0, B_1, B_2, \ldots, B_k \subseteq B$ be disjoint sets of vertices such that

- for $0 \le i, j \le k$, $i \neq j$, A_i is anticomplete to B_j;
- A_0 is anticomplete to B_0;
- for $i \in [k]$, $|A_i| \ge 1$, $|B_i| \ge 1$, and A_i is complete to B_i.

We may choose these sets such that k is maximal and, subject to that, such that their union is maximal. Notice that we allow A_0 and B_0 to be empty, but the sets A_i, B_i, $i \in [k]$, are nonempty. Notice also that, because $2 \in \ell(F)$, $k \ge 1$.

(iii) *$A = \bigcup_{i=0}^{k} A_i$ and $B = \bigcup_{i=0}^{k} B_i$.*

Suppose not. From the symmetry, we may assume that there exists $a \in A \setminus \bigcup_{i=0}^{k} A_i$. First, a has at least one neighbor in $\bigcup_{i=0}^{k} B_i$, because otherwise we may add a to A_0, contradicting the maximality of A_0. Suppose that a has neighbors $b_i \in B_i$, $b_j \in B_j$ with $0 \le i < j \le k$. Let $a_j \in A_j$. Now, $G|\{p_1, p_2, \ldots, p_m, a, b_i, a_j, b_j\}$ is isomorphic to $\mathcal{G}_1$ (if $m = 3$) or $\mathcal{G}_2$ (if $m = 5$), a contradiction. Thus, a has a neighbor in B_i for only one value of i. If a has a neighbor in B_0, then letting $A_{k+1} = \{a\}$ and $B_{k+1} = N(a) \cap B_0$ contradicts the maximality of k. Thus, a has a neighbor $b \in B_i$ for some $i \in [k]$. By the maximality of A_i, a has a nonneighbor $b' \in B_i$. Let $a' \in A_i$. Now, $G|\{p_1, p_2, \ldots, p_m, a', b, a, b'\}$ is isomorphic to $\mathcal{G}_1$ (if $m = 3$) or $\mathcal{G}_2$ (if $m = 5$), a contradiction. This proves (iii). □

Next, we analyze how vertices in D attach to $A \cup B$:

(iv) *For $i \in [k]$, if $d \in D$ has a neighbor in $A_i \cup B_i$, then d is complete to $A_i \cup B_i$ and anticomplete to $A \cup B \setminus (A_i \cup B_i)$.*

From the symmetry, we may assume that $d \in D$ has a neighbor $a \in A_i$. Let $b \in B_i$. Recall that A_i is complete to B_i. Hence, a is complete to $\{b, d, p_m\}$. It follows from (2.2) that d is adjacent to b. Thus, d is complete to B_i and, by the same argument, d is complete to A_i. It follows from (ii) that d is anticomplete to $A_j \cup B_j$ for $j \in [k] \cup \{0\}$, $j \neq i$. This proves (iv). □

(v) *There do not exist $d_1, d_2 \in D$ such that d_1 has a neighbor in A_0 and d_2 has a neighbor in B_0.*

Let $d_1 \in D$ have a neighbor $a_0 \in A_0$ and let $d_2 \in D$ have a neighbor $b_0 \in B_0$. It follows from (ii) and (iv) that d_1 is anticomplete to $(A \cup B) \setminus A_0$ and d_2 is anticomplete to $(A \cup B) \setminus B_0$. Let $a_1 \in A_1$, $b_1 \in B_1$. Then, a_0-d_1-d_2-b_0-b_1-a_1-a_0 is an induced cycle of length six, a contradiction. This proves (v). □

By (v) and the symmetry, we may assume that D is anticomplete to B_0. For $i \in [k] \cup \{0\}$, let D_i be the vertices in D that have a neighbor in $A_i \cup B_i$. It follows from (iv) that the sets $D_0, D_1, \ldots, D_k$ are disjoint and that, for $i \in [k]$, D_i is complete to $A_i \cup B_i$. It follows from (ii) that D_0 is anticomplete to B. Let $D^* = D \setminus (D_0 \cup D_1 \cup \cdots \cup D_k)$. We need one more lemma:

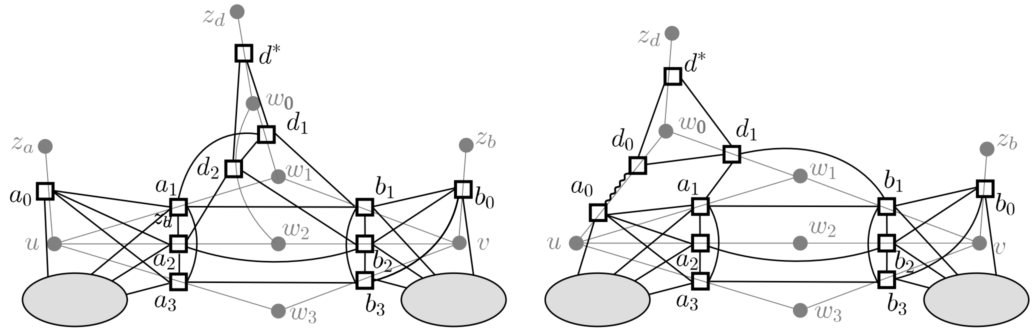
(vi) *There are at most two values $i \in [k] \cup \{0\}$ such that $D_i \neq \emptyset$.*

Suppose that there are i, j, l with $0 \le i < j < l \le k$ such that D_i, D_j and D_l are nonempty. It follows that A_i, A_l, B_j, B_l are all nonempty. Let $a_i \in A_i$, $a_l \in A_l$, $d_i \in D_i$, $d_j \in D_j$, $b_j \in B_j$ and $b_l \in B_l$ such that the pairs a_i, d_i and a_l, d_l are adjacent. Then, a_i-d_i-d_j-b_j-b_l-a_l-a_i is an induced cycle of length six, a contradiction. This proves (vi). □

We will construct a new representation (T'', H', η'); see Fig. 4 for an illustration of the construction. First construct T' from $T \setminus \eta(F)$ as follows. Let

$$K_1 = \bigcup\{\eta(F', u) \mid F' \in E(H) \setminus \{F\},\ u \in \bar{F}'\}, \quad \text{and}$$
$$K_2 = \bigcup\{\eta(F', v) \mid F' \in E(H) \setminus \{F\},\ v \in \bar{F}'\}.$$

Add a strong clique of new vertices $\bar{A} = \{a_0, a_1, \ldots, a_k\}$ such that $\bar{A}$ is strongly complete to K_1, add a strong clique of new vertices $\bar{B} = \{b_0, b_1, \ldots, b_k\}$ such that $\bar{B}$ is strongly complete to K_2, and add a strong clique of new vertices $\bar{D} = \{d_0, d_1, \ldots, d_k\}$. If $D^* \neq \emptyset$, then add a new vertex d^* that is strongly complete to $\bar{D}$. For $i \in [k]$, let $\{a_i, b_i, d_i\}$ be a strong clique. If A_0 is strongly complete to D_0, let a_0 be strongly adjacent to d_0; if A_0 is strongly anticomplete to D_0, let a_0 be strongly antiadjacent to d_0; otherwise let a_0 be semiadjacent to d_0. All other pairs are strongly antiadjacent. Let $X' \subseteq \{a_0, b_0\}$ be such that $a_0 \in X'$ if and only if $A_0 = \emptyset$ and $b_0 \in X'$ if and only if $B_0 = \emptyset$. Let $X = X' \cup \{d_i \mid D_i = \emptyset, i \in [k] \cup \{0\}\}$. Let $T'' = T' \setminus X$. Then, G is a graphic thickening of T''.



(a) $D_0 = \emptyset$. (b) $D_0 \neq \emptyset$.

Fig. 4. The construction of a larger strip-structure in (4.10). The gray vertices and edges represent the relevant submultigraph of H'. The black vertices and edges represent the relevant induced subtrigraph of T'. The gray ellipses represent the sets K_1 and K_2. The 'wiggly' edge represents a semiedge. The black vertices are drawn on top of the gray edges to indicate to which strip each black vertex belongs.

Next, construct (H', η') from (H, η) as follows. First, delete F. For $i \in [k]$, add new vertices w_i, and edges $F_{1,i}, F_{2,i}$ with $\bar{F}_{1,i} = \{u, w_i\}$ and $\bar{F}_{2,i} = \{v, w_i\}$, and let $\eta'(F_{1,i}) = \eta'(F_{1,i}, u) = \eta'(F_{1,i}, w_i) = \{a_i\}$ and $\eta'(F_{2,i}) = \eta'(F_{2,i}, v) = \eta'(F_{2,i}, w_i) = \{b_i\}$. If $B_0 \neq \emptyset$, then add a new vertex z_b and an edge F_b with $\bar{F}_b = \{v, z_b\}$ and $\eta'(F_b) = \eta'(F_b, v) = \eta'(F_b, z_b) = \{b_0\}$. Now, there are two cases, depending on whether D_0 is empty or not.

The case when D_0 is empty. If $A_0 \neq \emptyset$, then add a new vertex z_a and an edge F_a with $\bar{F}_a = \{u, z_a\}$ and $\eta'(F_a) = \eta'(F_a, u) = \eta'(F_a, z_a) = \{a_0\}$. It follows from (vi) and the symmetry that we may assume that $D_i = \emptyset$ for all $i \in [k] \setminus \{1, 2\}$. If $D = \emptyset$, then the construction of (T'', H', η') is complete. So we may also assume that $D_1 \neq \emptyset$. Add to H' a new vertex w_0. If $D^* \neq \emptyset$, then add a new vertex z_d and an edge F_d with $\bar{F}_d = \{w_0, z_d\}$ and $\eta'(F_d) = \eta'(F_d, w_0) = \eta'(F_d, z_d) = \{d^*\}$. For $i = 1, 2$, if $D_i \neq \emptyset$, then add a new edge F_i with $\bar{F}_i = \{w_i, w_0\}$, $\eta'(F_i) = \eta'(F_i, w_i) = \eta'(F_i, w_0) = \{d_i\}$. This finishes the construction of (T'', H', η') when $D_0 = \emptyset$ (see Fig. 4(a)).

The case when D_0 is nonempty. The fact that D_0 is nonempty implies that A_0 is nonempty. It follows from (vi) and the symmetry that we may assume that $D_i = \emptyset$ for all $i \in [k] \setminus \{1\}$. Add a new vertex w_0. If $D^* \neq \emptyset$, then add a new vertex z_d and an edge F_d with $\bar{F}_d = \{w_0, z_d\}$ and $\eta'(F_d) = \eta'(F_d, w_0) = \eta'(F_d, z_d) = \{d^*\}$. Add to H' a new edge F_0 with $\bar{F}_0 = \{u, w_0\}$, and $\eta'(F_0, u) = \{a_0\}$, $\eta'(F_0, w_0) = \{d_0\}$, and $\eta'(F_0) = \{a_0, d_0\}$. Notice that the strip of (H', η') at F_0 is a member of $\mathcal{Z}_1$. If $D_1 \neq \emptyset$, then add a new edge F_1 with $\bar{F}_1 = \{w_0, w_1\}$, $\eta'(F_1) = \eta'(F_1, w_0) = \eta'(F_1, w_1) = \{d_1\}$. This finishes the construction of (T'', H', η') when $D_0 \neq \emptyset$ (see Fig. 4(b)).

Now G is a graphic thickening of T'', T'' is not a thickening of any other claw-free trigraph, (H', η') is a proper strip-structure for T'', and $|E(H')| > |E(H)|$, contrary to the fact that (T, H, η) is an optimal representation for G. This proves (4.10). $\square$

(4.11). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let C be a cycle in H and let $F \in E(C)$ be such that $\ell(E(C \setminus F)) \cap \{3, 4, 5, 6\} \neq \emptyset$. Then, the strip of (H, η) at F is a spot.*

Proof. We may assume that $\ell(F) \neq \{1\}$. If $6 \in \ell(E(C \setminus F))$, then it follows from (4.7) that $\ell(F) = \{1\}$, contrary to our assumption. If $5 \in \ell(E(C \setminus F))$, then it follows from (4.7) that $\ell(F) = \{2\}$, contrary to (4.10). If $4 \in \ell(E(C \setminus F))$, then, since $\ell(F) \neq \{1\}$, it follows from (4.7) that $\ell(F) = \{3\}$, contrary to (4.9). Thus, we may assume that $3 \in \ell(E(C \setminus F))$. It follows from (4.7) that $\ell(F) \subseteq \{2, 4\}$. It follows from (4.10) that $\ell(F) \neq \{2\}$ and $\ell(F) \neq \{2, 4\}$. Thus, $\ell(F) = \{4\}$. But this contradicts (4.9). This proves (4.11). $\square$

Another useful corollary is the following description of possible strips in optimal representations:

(4.12). *Let (T, H, η) be an optimal representation of some $\mathcal{F}$-free claw-free graph G. Let $F \in E(H)$ with $\ell(F) \neq \emptyset$ and let $\{u, v\} = \bar{F}$. Then either*

(a) *the strip of (H, η) at F is a spot, or*
(b) *$\eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$ is a strong clique and $z \leq 4$ for all $z \in \ell(F)$, or*
(c) *the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_1$, $2 \notin \ell(F)$, and there exists $z \in \ell(F)$ with $z \geq 4$.*

Proof. We may assume that $1 \notin \ell(F)$, because otherwise, by (4.6), case (a) holds. Let (J, Z) be the strip of (H, η) at F. Since $\ell(F) \neq \emptyset$, it follows from (4.6) that (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$. If (J, Z) is isomorphic to a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$, then it follows from the definition of the respective strips that $\eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$ is a strong clique, and hence outcome (b) holds (the fact that $z \leq 4$ for all $z \in \ell(F)$ follows immediately). Therefore, we may assume that (J, Z) is isomorphic to a member of $\mathcal{Z}_1$, and thus J is a linear interval trigraph. Let $A = \eta(F, u)$, $B = \eta(F, v)$, and $C = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. If $2 \in \ell(F)$, then there exist adjacent $a \in A$ and $b \in B$, and hence it follows from the definition of a linear interval trigraph that C is a strong clique and thus (b) holds. So we may assume that $2 \notin \ell(F)$. It follows from (4.9) that $\ell(F) \neq \{3\}$ and therefore there exists $z \in \ell(F)$ with $z \geq 4$, which implies that outcome (c) holds. This proves (4.12). $\square$

4.2. The structure of the blocks of the pattern multigraph in an optimal representation

Let T be a connected claw-free trigraph that admits a proper strip-structure (H, η) such that H is not 2-connected. Let B be a block of H and let W be the cut-vertices of H in $V(B)$. Let D be the trigraph obtained from $T|\bigcup\{\eta(F) \mid F \in E(B)\}$ by adding, for every $w \in W$, a vertex y that is complete to $\bigcup\{\eta(F, w) \mid F \in E(B)\}$. Let Y be the vertices added in that way. We call (D, Y) the *strip-block of* (H, η) *at* B.

(4.13). *Let T be a connected $\mathcal{F}$-free claw-free trigraph that admits a proper strip-structure (H, η). Let $(B_1, B_2, \ldots, B_q)$ be the block decomposition of H. Then, at most one of $B_1, B_2, \ldots, B_q$ contains a cycle of length at least five.*

Proof. Suppose that for distinct $i \in [2]$, B_i contains a cycle of length $k_i \geq 5$. It follows from the definition of a proper strip-structure that for $i \in [2]$ the strip-block (D_i, X_i) of (H, η) at B_i contains a weakly induced cycle C_i with $|V(C_i)| \in \{5, 7\}$ (because T is $\mathcal{F}$-free). Because C_1 and C_2 are in different strip-blocks, it follows that $V(C_1) \cap V(C_2) = \emptyset$. Let $C_1 = c_1\text{-}\cdots\text{-}c_{k_1}\text{-}c_1$ and $C_2 = c'_1\text{-}\cdots\text{-}c'_{k_2}\text{-}c'_1$. Since T is connected, there exists a shortest weakly induced path $P = p_1\text{-}p_2\text{-}\cdots\text{-}p_m$ from a vertex in $V(C_1)$ to a vertex in $V(C_2)$. We may assume that $p_1 = c_1$ and $p_m = c'_1$. First suppose that $m = 2$. Because c_1 is complete to $\{c'_1, c_2, c_{k_1}\}$, it follows that c'_1 is adjacent to at least one of c_2, c_{k_1}. From the symmetry, we may assume that c'_1 is adjacent to c_2. Symmetrically, we may assume that c_1 is adjacent to c'_2. Since, by the definition of a strip-structure, $N(C_1) \cap V(C_2)$ and $N(C_2) \cap V(C_1)$ are strong cliques, it follows that c_1 is strongly anticomplete to $V(C_2) \setminus \{c'_1, c'_2\}$ and c'_1 is strongly anticomplete to $V(C_1) \setminus \{c_1, c_2\}$. If c_2 is antiadjacent to c'_2, then c'_1 is complete to the triad $\{c_2, c'_2, c'_{k_2}\}$, contrary to (2.2). Thus, c_2 is strongly adjacent to c'_2. Since $N(C_1) \cap V(C_2)$ and $N(C_2) \cap V(C_1)$ are strong cliques, it follows that $N(C_1) \cap V(C_2) = \{c'_1, c'_2\}$ and $N(C_2) \cap V(C_1) = \{c_1, c_2\}$. Thus, $T|(V(C_1) \cup V(C_2))$ is a weakly induced skipping rope, a contradiction. So we may assume that $m \geq 3$. Since P is shortest, it follows that $V(C_1) \cup V(P^*)$ is strongly anticomplete to $V(C_2)$ and $V(C_2) \cup V(P^*)$ is strongly anticomplete to $V(C_1)$. Because c_1 is complete to $\{p_2, c_2, c_{k_1}\}$, it follows from (2.2) that p_2 is adjacent to at least one of c_2, c_{k_1}. We may assume that p_2 is adjacent to c_2. Next, if p_2 is complete to antiadjacent $c, c' \in V(C_1)$, then p_2 is complete to the triad $\{p_3, c, c'\}$, contrary to (2.2). Hence, it follows that p_2 is strongly anticomplete to $V(C_1) \setminus \{c_1, c_2\}$. Symmetrically, we may assume that p_m is complete to $\{c'_1, c'_2\}$ and strongly anticomplete to $V(C_2) \setminus \{c'_1, c'_2\}$. But now, $T|(V(C_1) \cup V(C_2) \cup V(P))$ is a weakly induced skipping rope, a contradiction. This proves (4.13). □

As the previous lemma suggests, when we describe the blocks, it is convenient to distinguish between blocks that contain a cycle of length at least five, and blocks that do not contain such a cycle. We start with the former case. In [2], we implicitly proved the following result. For completeness we give the proof of it here.

(4.14). *Let H be a 2-connected simple graph with no cycle of length k with $k = 6$ or $k \geq 8$. Then, either every cycle in H has length at most 4, or H is isomorphic to a graph in $\mathcal{B}_1$.*

Proof. We use induction on $|E(H)|$. Let $F = f_1\text{-}f_2\text{-}\cdots\text{-}f_k\text{-}f_1$ be a largest cycle in H. If $k \leq 4$, then the lemma holds. Thus, since H has no cycle of length six or of length eight or more, we may assume that $k \in \{5, 7\}$. We say that a vertex $x \in V(H) \setminus V(F)$ is a *clone for F* if, for some $i \in [k]$, $N(x) \cap V(F) = \{f_{i-1}, f_{i+1}\}$ (subscript modulo k). In this case we say that x is a clone *of type i*. We start with a number of claims:

(i) *Every vertex in $V(H) \setminus V(F)$ is a clone for F.*

Let $x \in (V(H) \setminus V(F))$. Since H is 2-connected, there exist two paths P_1 and P_2 from x to two distinct vertices of F, say f_i and f_j, respectively, such that $V(P_1) \cap V(P_2) = \{x\}$. From the symmetry, we may assume that $i = 1$ and $j > k/2$. First assume that $|E(P_1)| + |E(P_2)| \geq 3$. Now $f_1\text{-}P_1^*\text{-}x\text{-}P_2^*\text{-}f_j\text{-}f_{j-1}\text{-}\cdots\text{-}f_2\text{-}f_1$ is a cycle of length $|E(P_1)| + |E(P_2)| + j - 1$ and $f_1\text{-}P_1^*\text{-}x\text{-}P_2^*\text{-}f_j\text{-}f_{j+1}\text{-}\cdots\text{-}f_k\text{-}f_1$ is a cycle of length $|E(P_1)| + |E(P_2)| + (k - j) - 1$. Thus, since H has no cycle of length six and by the maximality of F, we have

$$|E(P_1)| + |E(P_2)| + j - 1 \in [k] \setminus \{6\}, \quad \text{and} \quad |E(P_1)| + |E(P_2)| + (k - j) - 1 \in [k] \setminus \{6\}.$$

It is straightforward to check that this system has no solution if $|E(P_1)| + |E(P_2)| \geq 3$. It follows that $|E(P_1)| + |E(P_2)| = 2$ and, therefore, $|E(P_1)| = |E(P_2)| = 1$. Thus, x has two neighbors in $V(F)$. If x has two consecutive neighbors in $V(F)$, say f_1, f_2, then $f_1\text{-}x\text{-}f_2\text{-}f_3\text{-}\cdots\text{-}f_{k-1}\text{-}f_k\text{-}f_1$ is a cycle of length $k + 1$, contrary to the maximality of F. If $k = 5$, then, since x has at least two neighbors in $V(F)$, it follows that x is a clone for F. So we may assume that $k = 7$. Suppose that x is adjacent to f_p and f_{p+3} for some $p \in [7]$. From the symmetry, we may assume that $p = 1$. But now $f_1\text{-}x\text{-}f_4\text{-}f_5\text{-}f_6\text{-}f_7\text{-}f_1$ is a cycle of length six, a contradiction. From the symmetry, it follows that x has exactly two neighbors in F, say f_q and f_{q+2} for some $q \in [7]$. Hence, x is a clone for F. This proves (i). □

(ii) *If $x \in V(H) \setminus V(F)$ is a clone for F of type i, then no vertex in $V(H) \setminus V(F)$ is a clone of type $i + 1$* (modulo k).

From the symmetry, we may assume that x is a clone for F of type 1 and there exists $y \in V(H) \setminus V(F)$ that is a clone for F of type 2. Now, $f_1\text{-}f_k\text{-}x\text{-}f_2\text{-}f_3\text{-}y\text{-}f_1$ is a cycle of length six, a contradiction. This proves (ii).

(iii) *$V(H) \setminus V(F)$ is a stable set.*

Suppose that $x, y \in V(H) \setminus V(F)$ are adjacent. From (i), we may assume that x is a clone of type 1. From the symmetry and (ii), we may assume that y is a clone of type 1, type 3, or, if $k = 7$, of type 4. First suppose that y is a clone of type 1. Then $y\text{-}x\text{-}f_2\text{-}\cdots\text{-}f_k\text{-}y$ is a cycle of length $k + 1$, contrary to the maximality of F. Next, suppose that y is a clone of type 3.

Then, f_1-f_2-x-y-f_4-$\cdots$-f_k-f_1 is a cycle of length $k+1$, contrary to the maximality of F. Finally, suppose that $k=7$ and y is a clone of type 4. Then f_2-f_3-f_4-f_5-y-x-f_2 is a cycle of length six, a contradiction. This proves (iii). □

Now suppose that there exists $x \in V(H) \setminus V(F)$. It follows from (i) that x is a clone for F. From the symmetry, we may assume that x is a clone of type 1. We claim that $\deg(f_1) = 2$. For suppose not. Then f_1 has a neighbor $y \in V(H) \setminus \{f_k, f_1, f_2\}$. First suppose that $y \in V(H) \setminus V(F)$. It follows from (i) that y is a clone of type 2 or type k, contrary to (ii). Thus, it follows that $y = f_j$ for some $j \in \{3, \ldots, k-1\}$. From the symmetry, we may assume that either $j = 3$, or $k = 7$ and $j = 4$. First assume that $j = 3$. Then x-f_2-f_1-f_3-$\ldots$$f_k$-$x$ is a cycle of length $k+1$, a contradiction. So we may assume that $k = 7$ and $j = 4$. But now f_1-f_4-f_3-f_2-x-f_7-f_1 is a cycle of length six, a contradiction. This proves that $\deg(f_2) = 2$. Thus, H is obtained from $H \setminus \{x\}$ by cloning a vertex of degree two. Hence it follows from the induction hypothesis that H is isomorphic to a graph in $\mathcal{B}_1$ and therefore the lemma holds.

So we may assume that $V(H) = V(F)$. If $k = 5$, then H is isomorphic to a graph in $\mathcal{B}_1$ and the lemma holds. Therefore, we may assume that $k = 7$.

(iv) *Let $i \in [7]$. Then, f_i is nonadjacent to f_{i+2}.*

From the symmetry, we may assume that $i = 1$. If f_1 is adjacent to f_3, it follows that f_1-f_3-f_4-f_5-f_6-f_7-f_1 is a cycle of length six, a contradiction. This proves (iv). □

(v) *Let $i \in [7]$. If f_i is adjacent to f_{i+3}, then f_{i+5} is anticomplete to $\{f_{i+1}, f_{i+2}\}$.*

From the symmetry, we may assume that $i = 1$. Suppose that f_1 is adjacent to f_4. If f_6 is adjacent to f_2, then it follows that f_1-f_4-f_3-f_2-f_6-f_7-f_1 is a cycle of length six, a contradiction. This proves that f_6 is nonadjacent to f_2 and, symmetrically, f_6 is nonadjacent to f_3. This proves (v). □

If F is an induced cycle, then the lemma holds. Therefore, it follows from (iv) and the symmetry that we may assume that f_1 is adjacent to f_4. It follows from (v) that f_6 is anticomplete to $\{f_2, f_3\}$. First suppose that f_2 is nonadjacent to f_5 and f_3 is nonadjacent to f_7. Then, the only undetermined adjacencies are between the pairs f_4, f_7 and f_1, f_5. Hence, H is of the $\mathcal{B}_1$ type and the lemma holds. Therefore, we may assume from the symmetry that f_2 is adjacent to f_5. It follows from (v) that f_7 is anticomplete to $\{f_3, f_4\}$. Now the only undetermined adjacency is between f_1 and f_5. Thus, H is of the $\mathcal{B}_1$ type. This proves (4.14). □

Lemma (4.14) deals with blocks that contain a long cycle. For blocks with no such cycle, we use the following result from [6].

Theorem 4.15 ([6]). *Let G be a graph. Then, the following statements are equivalent:*

(1) *G does not contain any odd cycle of length at least 5.*

(2) *For every connected subgraph G' of G, either G' is isomorphic to K_4, or G' is a bipartite graph, or G' is isomorphic to $K_{2,t}^{+}$ for some $t \geq 1$, or G' has a cut-vertex.*

This allows us to prove the following structural description of blocks that do not contain cycles of length at least five.

(4.16). *Let H be a 2-connected graph with $|V(H)| \geq 2$ that contains no cycle of length five or longer. Then, H is isomorphic to a graph in $\mathcal{B}_2$.*

Proof. It follows from Theorem 4.15 that either H is isomorphic to K_4, or H is a bipartite graph, or H isomorphic to $K_{2,t}^{+}$ for some $t \geq 1$. If H is isomorphic to K_4, then H is of the $\mathcal{B}_2$ type. If H is isomorphic to $K_{2,t}^{+}$ for some $t \geq 1$, then H is either isomorphic to K_3 (if $t = 1$), or to $K_{2,t}^{+}$ with $t \geq 2$, both of which imply that H is of the $\mathcal{B}_2$ type. Therefore, we may assume that H is a bipartite graph. Let $V(H) = X \cup Y$ such that X and Y are stable sets. The 2-connectedness of H implies that $|X| \geq 2$ and $|Y| \geq 2$. Now suppose that $x \in X$ is nonadjacent to $y \in Y$. Since H is 2-connected, it follows that there are two edge-disjoint paths P_1 and P_2 from x to y. Since x and y are nonadjacent and H is bipartite, it follows that $|E(P_1)| \geq 3$ and $|E(P_2)| \geq 3$. But now x-P_1^*-y-P_2^*-x is a cycle of length at least six, a contradiction. It follows that X is complete to Y. If $|X| \geq 3$ and $|Y| \geq 3$, then clearly, H contains a cycle of length six, a contradiction. Therefore, at least one of X, Y has size exactly 2. Hence, H is isomorphic to $K_{2,t}$ with $t = \max\{|X|, |Y|\}$ and H is of the $\mathcal{B}_2$ type. This proves (4.16). □

This allows us to prove (4.2):

Proof of (4.2). Let G be a nonbasic connected $\mathcal{F}$-free claw-free graph. It follows from (2.3) that G is a graphic thickening of a claw-free trigraph T that admits a proper strip-structure. We may assume that T is not a thickening of some other trigraph. By choosing among all strip-structures of T, a strip-structure (H, η) of T that has a maximum number of edges, it follows that G has an optimal representation (T, H, η). Property (iii) follows from the following claim:

(∗) *Let C be a cycle in H with $|E(C)| \geq 4$. Then, $\ell(F) = \{1\}$ for all $F \in E(C)$.*

Let $F \in E(C)$. Since each edge in $E(C \setminus F)$ lies in a cycle, it follows that $\ell(F') \neq \emptyset$ for all $F' \in E(C \setminus F)$ and hence $z \geq 3$ for all $z \in \ell(E(C \setminus F))$. It follows from (4.7) that $z \leq 6$ for all $z \in \ell(E(C \setminus F))$. Since $\ell(E(C \setminus F))$ is nonempty, it follows that $\ell(E(C \setminus F)) \cap \{3, 4, 5, 6\} \neq \emptyset$ and, thus, by (4.11), $\ell(F) = \{1\}$.

By (4.4), T is $\mathcal{F}$-free. It follows from the fact that T is $\mathcal{F}$-free that H has no cycles of length six or of length at least eight. Let $B_1, B_2, \ldots, B_q$ be the block-decomposition of H. Consider B_i. We claim that B_i is either of the $\mathcal{B}_1$ type, or of the $\mathcal{B}_2$ type. If B_i contains a cycle of length at least five, then it follows from (4.14) that B_i is of the $\mathcal{B}_1$ type. So we may assume that B_i has no cycle of length at least five. Now, it follows from (4.16) applied to $U(B_i)$ that B_i is of the $\mathcal{B}_2$ type. This proves part (i). Finally, for part (ii), it follows from (4.13) and the fact that every block of the $\mathcal{B}_1$ type contains a cycle of length five or seven, that at most one block of H is of the $\mathcal{B}_1$ type. This proves (4.2). □

5. $\mathcal{F}$-free nonbasic trigraphs with stability number at most 3

Recall that, by (2.9), all $\mathcal{F}$-free claw-free trigraphs with stability number at most 2 are resolved. In this section, we deal with nonbasic $\mathcal{F}$-free claw-free trigraphs with stability number 3.

Let T be a trigraph. Assuming that T is connected, a strong clique X in T is called a *clique cutset* if $T - X$ is disconnected. Suppose that V_0, V_1, V_2 are disjoint sets with union $V(T)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

(1) $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (A_i \cup B_i)$ for $i = 1, 2$,
(2) for $i = 1, 2$, $A_i \cap B_i = \emptyset$, and A_i, B_i are nonempty, and
(3) for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$, and
(4) for $i = 1, 2$, $V_i \setminus (A_i \cup B_i)$ is nonempty.

We call the triple (V_0, V_1, V_2) a *generalized* 2*-join*. We call a triple (V_0, V_1, V_2) a *modified generalized* 2*-join* if (V_0, V_1, V_2) and A_1, A_2, B_1, B_2 satisfy properties (1)–(3) and, instead of (4), the following:

(4') for $i = 1, 2$, either $V_i \setminus (A_i \cup B_i)$ is nonempty, or $|A_i| = |B_i| = 1$ and the unique two vertices in $A_i \cup B_i$ are semiadjacent.

Because the trigraphs that we are interested in are nonbasic, they admit a proper strip-structure. The following lemma shows that such a trigraph is either the line graph of a 2-connected graph, or has a clique cutset, or admits a modified generalized 2-join.

(5.1). *Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Then one of the following three statements hold:*

(a) *all strips of (H, η) are spots and H is* 2*-connected, or*
(b) *T has a clique cutset, or*
(c) *T admits a modified generalized* 2*-join.*

Proof. We start with the case in which T, regarded as a graph, is a line graph:

(i) *If all strips of (H, η) are spots, then the lemma holds.*

Suppose that all strips of (H, η) are spots. If H is 2-connected, then outcome (a) holds. So we may assume that H has a cut vertex x. Let $X_1, \dots, X_q$ be the connected components of $H - x$ $(q \geq 2)$. Because T is nonbasic, H is not a star and, hence, there exists $i \in [q]$ such that X_i is not a single vertex. Because X_i is connected, X_i contains at least one edge. Now, $\{\eta(F) \mid \bar{F} = \{x, u\}, u \in V(X_i)\}$ is a clique cutset in T and (b) holds. This proves (i). □

By (i), we may assume that there exists $F^* \in E(H)$ such that the strip of (H, η) at F^* is not a spot. Let $\{u, v\} = \bar{F}^*$. First suppose that one of $\eta(F^*, u)$ and $\eta(F^*, v)$ is empty. We may assume that $\eta(F^*, v) = \emptyset$. Observe that $\eta(F^*) \neq \eta(F^*, u)$ since the strip of (H, η) at F^* is not a spot. Thus, $\eta(F^*, u)$ is a clique cutset and outcome (b) holds.

So we may assume that both $\eta(F^*, u)$ and $\eta(F^*, v)$ are nonempty. This implies that (J, Z) is isomorphic to a member of $\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_5$. Let $E_0 \subseteq E(H)$ be the set of edges F_0 such that $\bar{F}_0 = \bar{F}^*$ and the strip of (H, η) at F_0 is a spot. Notice that it follows from Observation 4.1 that $|E_0| \leq 1$. Let $E_2 = E(H) \setminus (E_0 \cup \{F^*\})$.

(ii) *E_2 is nonempty*.

Suppose that E_2 is empty. Then, since (H, η) is proper, we have that $|E_0| = 1$. Let F be the unique element of E_0. If (J, Z) is isomorphic to a member of $\mathcal{Z}_1$, then T is a circular interval trigraph. It follows that either T is three-cliqued, or T is a long circular interval trigraph. In either case, T is basic, a contradiction. Therefore, (J, Z) is isomorphic to a member of $\mathcal{Z}_2 \cup \cdots \cup \mathcal{Z}_5$ and hence (J, Z) is three-cliqued. But now, T is three-cliqued because $V(T)$ is the union of the strong cliques $\eta(F^*, u) \cup \eta(F)$, $\eta(F^*, v)$, and $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$, a contradiction. This proves (ii). □

Let $V_1 = \eta(F^*)$, $A_1 = \eta(F^*, u)$ and $B_1 = \eta(F^*, v)$. Next, set $V_0 = \bigcup\{\eta(F_0) \mid F_0 \in E_0\}$, $V_2 = \bigcup\{\eta(F_2) \mid F_2 \in E_2\}$, $A_2 = \bigcup\{\eta(F_2, u) \mid F_2 \in E_2, u \in \bar{F}_2\}$ and $B_2 = \bigcup\{\eta(F_2, v) \mid F_2 \in E_2, v \in \bar{F}_2\}$. Notice that $A_1 \cup A_2 \cup V_0$ and $B_1 \cup B_2 \cup V_0$ are strong cliques, and the sets A_2 and B_2 are disjoint. Since $E_2 \neq \emptyset$ by (ii) and H is connected (because T is connected), at least one of A_2, B_2 is nonempty. If one of A_2 is empty, then $B_1 \cup V_0$ is a clique cutset (separating A_1 from B_2) and outcome (b) holds. Thus, from the symmetry, we may assume that A_2, B_2 are both nonempty.

(iii) *For $i = 1, 2$, if $V_i = A_i \cup B_i$, then $|A_i| = |B_i| = 1$ and the unique two vertices in V_i are semiadjacent*.

Suppose that $V_i = A_i \cup B_i$. First suppose that one of A_i, B_i contains more than one vertex. Let j be such that $\{i, j\} = \{1, 2\}$. Let T' be the graph constructed from T as follows. First, replace A_i by a new vertex a_i and replace B_i by a new vertex b_i. Second, make a_i strongly complete to $A_j \cup V_0$ and b_j strongly complete to $B_j \cup V_0$. Third, depending on whether the pair (A_i, B_i) is strongly complete, strongly anticomplete, or neither, make a_i and b_i strongly adjacent, strongly antiadjacent, or semiadjacent, respectively. Now T is a thickening of T', a contradiction. This proves that $|A_i| = |B_i| = 1$. Let $a_i \in A_i$ and $b_i \in B_i$. If a_i is strongly adjacent to b_i, then we may enlarge the strip-structure (H, η) for T by replacing edge uv (in H) by a two-edge path (with a new internal vertex) between u, v and making the corresponding strips spots, thereby contradicting the optimality of the representation. If a_i is strongly antiadjacent to b_i, then we may enlarge the strip-structure (H, η) for T by replacing edge uv (in H) by a pending edge attached to each of u, v and making the corresponding strips spots, thereby contradicting the optimality of the representation. Thus, a_i and b_i are semiadjacent. This proves (iii). □

Now it follows from (iii) that (V_0, V_1, V_2) is a modified generalized 2-join and, thus, outcome (c) holds. This proves (5.1). □

The first lemma deals with the case where all strips are spots. That is, we deal with outcome (a) of (5.1).

(5.2). *Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Suppose that $\alpha(T) = 3$, all strips of (H, η) are spots and H is 2-connected. Then T is resolved.*

Proof. Notice that T is the line graph of H. It follows from (4.2) that H is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type. If H is of the $\mathcal{B}_2$ type, then every matching of H has size at most two, and hence $\alpha(T) \leq 2$, a contradiction. Thus, H is of the $\mathcal{B}_1$ type. First suppose that H contains a cycle of length 7. If $|V(H)| \geq 8$, then, by the construction of $\mathcal{B}_1$, H contains a matching of size four, and hence $\alpha(T) = 4$, a contradiction. Thus, H is isomorphic to a graph on vertex set $c_1, \dots, c_7$ where c_1-c_2-$\cdots c_7$-c_1 is a cycle and in which all pairs that are not in the cycle are nonadjacent except possibly a subset of the pairs $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_4, c_7\}$. It is easy to see that every maximal matching in this graph has size exactly 3. This implies that every maximal stable set in T has size 3, which means that T is resolved. So we may assume that H contains no cycle of length 7. Hence, H can be constructed, by nonadjacent cloning of vertices of degree 2, from a graph on 5 vertices that contains a cycle of length 5, say c_1-c_2-$\cdots$-c_5-c_1. If $|V(H)| = 5$, then every maximal matching in H has size two, and thus T is resolved. So we may assume that $|V(H)| \geq 6$. By the symmetry, we may assume that the vertices of degree 2 that were cloned form a nonempty subset of $\{c_1, c_3\}$. Thus, $c_1c_3 \notin E(H)$. Now notice that $\delta(c_2) \cup \delta(c_4) \cup \delta(c_5) = E(H)$ (where $\delta(u)$ is the set of edges of H that are incident with vertex u). This implies that T is three-cliqued, contrary to the assumption that T is nonbasic. This proves (5.2). □

The next lemma deals with clique cutsets (i.e., outcome (b) of (5.1)).

(5.3). *Let T be an $\mathcal{F}$-free nonbasic claw-free trigraph with $\alpha(T) = 3$. If T has a clique cutset, then T is resolved.*

Proof. Let X be a clique cutset in T. Let $K_1, K_2, \dots, K_m$ be the connected components of $T \setminus X$. Since X is a clique cutset, $m \geq 2$. Because $\alpha(T) \leq 3$, it follows that for all i, j, at least one of K_i, K_j is a strong clique. Therefore, there exists i such that K_i is a strong clique. Now it follows from (2.7) applied to K_i and X that T is resolved. This proves (5.3). □

The following lemma deals with modified generalized 2-joins (i.e., outcome (c) of (5.1)).

(5.4). *Let T be an $\mathcal{F}$-free nonbasic claw-free trigraph with $\alpha(T) = 3$. Suppose that T admits a modified generalized 2-join. Then, T is resolved.*

Proof. For $i = 1, 2$, let V_i, A_i, B_i and V_0 be as in the definition of a modified generalized 2-join. Let $Q_i = V_i \setminus (A_i \cup B_i)$. In view of (5.3), we may assume that T has no clique cutset.

First suppose that $Q_1 = \emptyset$. Property (4') of a modified generalized 2-join implies that $|A_1| = |B_1| = 1$ and the unique two vertices of $A_1 \cup B_1$ are semiadjacent. Since $\alpha(T) = 3$, it follows that Q_2 is a strong clique. But now, $A_1 \cup V_0 \cup B_1, A_2 \cup B_2$ and Q_2 are strong cliques, which implies that T is three-cliqued, contrary to our assumption that T is nonbasic. Thus, we may assume that Q_1 and, by the symmetry, Q_2 are nonempty.

If, for some $i \in [2]$, A_i is strongly complete to B_i, then $A_i \cup B_i$ is a clique cutset in T, a contradiction. Hence, for $i \in [2]$, A_i is not strongly complete to B_i. Next, it follows from the fact that $\alpha(T) = 3$ and $Q_1, Q_2 \neq \emptyset$, that $\alpha(T|V_i) \leq 2$ for $i = 1, 2$. Let $\{i, j\} = \{1, 2\}$ and suppose that Q_i is not a strong clique. Since $\alpha(T) = 3$, it follows that V_j is a strong clique and hence that A_j is strongly complete to B_j, a contradiction. Thus, Q_1 and Q_2 are strong cliques.

Let $i \in [2]$. If $N(Q_i)$ is a strong clique, then $N(Q_i)$ is a clique cutset, a contradiction. It follows that there exist antiadjacent $a_i, b_i \in N(Q_i)$ and, because A_i and B_i are strong cliques, we may assume that $a_i \in A_i$ and $b_i \in B_i$. It follows that there exist $p_i, q_i \in Q_i$ (possibly equal) such that p_i is adjacent to a_i and q_i is adjacent to b_i. Since T has no weakly induced cycles of length six or of length at least 8, it follows that we may assume that $p_1 \neq q_1$, p_1 is strongly antiadjacent to b_1, q_1 is strongly antiadjacent to a_1, and $p_2 = q_2$. Since T has no weakly induced cycle of length six, it follows that A_2 is strongly anticomplete to B_2. Moreover, since $\alpha(T) = 3$, it follows from the fact that p_1 is antiadjacent to b_1 that Q_2 is strongly complete to A_2 and hence, from the symmetry, that Q_2 is strongly complete to B_2.

Let G be an $\mathcal{F}$-free graphic thickening of T. We claim that G is resolved. For $v \in V(T)$, let X_v be the clique in G corresponding to v. For $i \in [2]$, let $V_i' = \bigcup\{X_v \mid v \in V_i\}$ and define A_i', B_i', Q_i', and V_0' analogously. Observe that T contains a weakly induced cycle of length seven. Therefore, by (2.1) and the strong perfect graph theorem [4], G is not perfect. Thus, if every maximal stable set in G has size three, then G satisfies condition (c) of the definition of a resolved graph and hence G is resolved. Clearly, no vertex is complete to all other vertices, so there is no maximal stable set of size one. So we may assume that there exists a maximal stable set $S = \{s_1, s_2\}$ of size two in G. If $S \cap V_2' = \emptyset$, then we may add any vertex from Q_2' to S to obtain a larger stable set, a contradiction. If $S \subseteq V_2'$, then we may add any vertex from Q_1' to S to obtain a larger stable set, a contradiction. It follows that $|S \cap V_2'| = 1$ and hence $|S \cap (V_0' \cup V_1')| = 1$. We may assume that $s_1 \in V_0' \cup V_1'$ and $s_2 \in V_2'$. If $s_1 \in V_0'$, then we may add any vertex from Q_1' to S to obtain a larger stable set, a contradiction. It follows that $s_1 \in V_1'$. We need the following observation:

($*$) *If $q_1' \in Q_1'$ has neighbors $a_1' \in A_1'$ and $b_1' \in B_1'$, then a_1' is adjacent to b_1'.*

Suppose not. Then, let $a_2' \in A_2'$, $b_2' \in B_2'$, and $q_2' \in Q_2'$ and observe that a_1'-q_1'-b_1'-b_2'-q_2'-a_2'-a_1' is an induced cycle of length six, a contradiction. This proves ($*$). □

First suppose that $s_1 \in Q_1'$. Since A_1' is not complete to B_1', there exist nonadjacent $a_1' \in A_1'$ and $b_1' \in B_1'$. It follows from $(*)$ that s_1 is not complete to $\{a_1', b_1'\}$. From the symmetry, we may assume that s_1 is nonadjacent to a_1'. It follows from the maximality of S that $s_2 \in A_2$. But now, we may add any vertex from B_2' is S to obtain a larger stable set, a contradiction. This proves that $s_1 \notin Q_1'$. Therefore, from the symmetry, we may assume that $s_1 \in A_1'$. The maximality of S implies that s_1 is complete to Q_1'. In particular, s_1 is complete to X_{p_1} and X_{q_1}. Since X_{q_1} is complete to X_{b_1}, it follows from $(*)$ that s_1 is complete to X_{b_1}. But now, s_1 is complete to the triad $\{a_2', b_1', p_1'\}$ with $a_2' \in A_2'$, $b_1' \in X_{b_1}$ and $p_1' \in X_{p_1}$, contrary to (2.2). This proves that G is resolved, which implies that T is resolved, thus proving (5.4). □

This leads to the main result of this subsection:

(5.5). *Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. If $\alpha(T) \leq 3$, then T is resolved.*

Proof. If $\alpha(T) \leq 2$, then it follows from (2.9) that T is resolved. Thus, we may assume that $\alpha(T) = 3$. Since T is nonbasic, T has an proper strip-structure (H, η). It follows from (5.1) that either H is 2-connected and T is the line graph of H, or T has a clique cutset, or T admits a modified generalized 2-join. Hence, it follows from (5.2), (5.3) and (5.4) that T is resolved. This proves (5.5). □

6. $\mathcal{F}$-free nonbasic claw-free graphs are resolved

We are now ready to prove that nonbasic $\mathcal{F}$-free claw-free graphs are resolved. In Section 5, we dealt with nonbasic trigraphs that have stability number at most three, so we may assume that our trigraphs have stability number at least four. In view of the definition of a (tri)graph being resolved, this means that we always look for dominant cliques. In Section 3, we gave a structure theorem for the pattern multigraph H for an optimal representation (T, H, η) of an $\mathcal{F}$-free nonbasic claw-free trigraph and we stated this structure in terms of the block decomposition of H. After introducing a few more lemmas in Section 6.1, we will deal, in Section 6.2, with trigraphs for which the pattern multigraph of an optimal representation is 2-connected. Then, in Section 6.3, we will deal with trigraphs whose pattern multigraph in an optimal representation is not 2-connected.

6.1. Tools

We need a few more tools that help us conclude that graphs are resolved. We need the following result on clones of vertices of degree 2.

(6.1). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Suppose that there exist $x_1, x_2 \in V(H)$ with $N(x_1) = N(x_2) = \{u, v\}$ such that the strips of (H, η) at F with $\bar{F} \in \{\{u, x_1\}, \{v, x_1\}, \{u, x_2\}, \{v, x_2\}\}$ are all spots. Then, G is resolved.*

Proof. Let E_u be the set of edges in H incident with u. Let $K = \bigcup\{\eta(F, u) \mid F \in E_u\}$. We claim that K is a dominant clique in T. For suppose not. Then, there exists a stable set $S \subseteq V(T) \setminus K$ that covers K. For $i = 1, 2$, let $z_i \in \eta(ux_i)$. For $i \in \{1, 2\}$, since $z_i \notin S$ and S covers K, it follows that there exist $y_i \in S$ that is adjacent to z_i. It follows from the assumptions and the choice of K that $y_i \in \eta(vx_i)$. But now it follows that y_1 and y_2 are strongly adjacent, contrary to the fact that S is a stable set. This proves (6.1). □

(6.2). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let $F \in E(H)$ and let $\{u, v\} = \bar{F}$. If $\ell(F) = \{2\}$, then either $\eta(F) = \eta(F, u) \cup \eta(F, v)$, or T is resolved.*

Proof. Let $A = \eta(F, u)$, $B = \eta(F, v)$, $C = \eta(F) \setminus (\eta(F, u) \cup \eta(F, v))$. We may assume that $C \neq \emptyset$, because otherwise the lemma holds. Since $2 \in \ell(F)$, it follows from (4.12) that C is a strong clique. If $N(C)$ is a strong clique, then (2.7) applied to $N(C)$ and C implies that G is resolved, and the lemma holds. Thus, we may assume that $N(C)$ is not a strong clique. Therefore, since A, B are strong cliques and $N(C) \subseteq A \cup B$, there exist antiadjacent $a \in A \cap N(C)$, $b \in B \cap N(C)$ and a weakly induced path P from a to b with $V(P^*) \subseteq C$ and $|V(P)| \in \{3, 4\}$. But this implies that $|V(P)| \in \ell(F)$, a contradiction. This proves (6.2). □

6.2. 2-connected strip-structures

We start with trigraphs whose pattern multigraph in the optimal representation is 2-connected.

(6.3). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. If H is 2-connected, then G is resolved.*

Proof. In view of (5.5), we may assume that $\alpha(T) \geq 4$. It follows from (4.2) and the fact that H is 2-connected that H is either of the $\mathcal{B}_1$ type or of the $\mathcal{B}_2$ type.

First suppose that H is of the $\mathcal{B}_1$ type. Since every edge of H is in a cycle of length 4, 5, or 7, it follows from (4.2) that $\ell(F) = \{1\}$ for all $F \in E(H)$. If there exist $F_1, F_2 \in E(H)$ with $\bar{F}_1 = \bar{F}_2$ then $\{F_1, F_2\}$ is a cycle that contradicts (4.7). Thus, H has no parallel edges. It follows that T, regarded as a graph, is the line graph of H. If H contains nonadjacent clones of vertices of degree 2, then it follows from (6.1) that G is resolved. So we may assume that H contains no such clones, and thus $U(H)$ is isomorphic to a graph in $\mathcal{B}_1^*$. But now, it is straightforward to check that $\alpha(T) \le 3$, a contradiction.

So we may assume that H is of the $\mathcal{B}_2$ type. It follows that $U(H)$ is either isomorphic to K_m for some $m \in \{2, 3, 4\}$, or to $K_{2,t}$ or $K_{2,t}^+$ for some $t \ge 2$. We prove the lemma by considering each case separately.

(i) *If $U(H)$ is isomorphic to K_2, then there is no F^* with $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$.*

Suppose that such F^* exists. Let u, v be the unique two vertices of H. It follows from the fact that (H, η) is proper that $|E(H)| \ge 2$. Clearly, if all strips of (H, η) are spots, then $\alpha(T) = 1$, a contradiction. Thus, the strip of (H, η) at F^* is not a spot. First suppose that $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$ is a strong clique. Then, T is the union of three strong cliques $\bigcup\{\eta(F, u) \mid F \in E(H)\}$, $\bigcup\{\eta(F, v) \mid F \in E(H)\}$, and $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$, and thus $\alpha(T) \le 3$, a contradiction. Thus, $\eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$ is not a strong clique. It follows from (4.12) that the strip of (H, η) at F^* is in $\mathcal{Z}_1$ and that $2 \notin \ell(F^*)$. Now, T is a long circular interval trigraph, a contradiction. This proves (i). □

(ii) *If $U(H)$ is isomorphic to K_2, then G is resolved.*

It follows from the fact that (H, η) is proper that $|E(H)| \ge 2$. Let z be maximum such that $z \in \ell(F^*)$ for some $F^* \in E(H)$. It follows from (4.7) that $z \le 6$, and it follows from (i) that $z \ge 2$. Let $\{u, v\} = \bar{F}^* = V(H)$. Now there are five cases.

First suppose that $z = 6$. It follows that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to (i). Next, suppose that $z = 5$. Let $F \in E(H) \setminus \{F^*\}$. It follows from (4.7) that $\ell(F) = \{2\}$, contrary to (4.10). Next, suppose that $z = 4$. It follows from (4.7) that $\ell(F) \in \{\{1\}, \{3\}\}$ for all $F \in E(H) \setminus \{F^*\}$. Since, by (4.9), no $F \in E(H) \setminus \{F^*\}$ satisfies $\ell(F) = \{3\}$, it follows that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to (i). Now, suppose that $z = 3$. It follows from (4.7) that either $\ell(F) = \{1\}$ or $\ell(F) = \{2\}$ for all $F \in E(H) \setminus \{F^*\}$. It follows from (4.10) that $\ell(F) \not\subseteq \{2, 4\}$ for all $F \in E(H) \setminus \{F^*\}$. Therefore, $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$, contrary to (i). So we may assume that $z = 2$. It follows from (4.12) that for every $F \in E(H)$ with $\ell(F) = \{2\}$, $\eta(F) = \eta(F, u) \cup \eta(F, v)$. Hence, T is the union of two strong cliques, namely $\bigcup\{\eta(F, u) \mid F \in E(H)\}$ and $\bigcup\{\eta(F, v) \mid F \in E(H), \ell(F) = \{2\}\}$. Therefore, $\alpha(T) \le 2$, a contradiction. This proves (ii). □

(iii) *If $U(H)$ is isomorphic to K_3, then G is resolved.*

Let z be maximum such that $z \in \ell(F^*)$ for some $F^* \in E(H)$. It follows from (4.7) that $z \le 5$. Let $V(H) = \{c_1, c_2, c_3\}$ such that $\bar{F}^* = \{c_1, c_2\}$. Now, there are five cases.

First suppose that $z = 5$. Then, by (4.7), $\ell(F) = \{1\}$ for all $F \in E(H)$ such that $\bar{F} \ne \{c_1, c_2\}$. If there exists $F \in E(H) \setminus \{F^*\}$ such that $\bar{F} = \{c_1, c_2\}$, then it follows from (4.7) that $\ell(F) = \{2\}$, contrary to (4.10). Thus, no such F exists. It follows from (4.12) that the strip of (H, η) at F^* is in $\mathcal{Z}_1$ and that $2 \notin \ell(F^*)$. But now, T is a long circular interval trigraph, a contradiction. Next, suppose that $z = 4$. Let $F_1, F_2 \in E(H)$ be such that $\bar{F}_1 = \{c_1, c_3\}$ and $\bar{F}_2 = \{c_2, c_3\}$. It follows from (4.7) that exactly one of F_1, F_2, say F', satisfies $\ell(F') = \{2\}$. But now consider $C = \{F^*, F_1, F_2\}$. It follows that $5 \in \ell(E(C) \setminus F')$, contrary to (4.10). Now, suppose that $z = 3$. It follows from (4.9) that $\ell(F^*) = \{2, 3\}$. Therefore, it follows from (4.10) that $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} \ne \{c_1, c_2\}$. Moreover, it follows from (4.7) and (4.10) that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$ with $\bar{F} = \{c_1, c_2\}$. It follows from (4.12) that $\eta(F^*) \setminus (\eta(F^*, c_1) \cup \eta(F^*, c_2))$ is a strong clique. Now, T is the union of three strong cliques $\bigcup\{\eta(F, c_1) \mid F \in E(H)\}$, $\bigcup\{\eta(F, c_2) \mid F \in E(H)\}$, and $\eta(F^*) \setminus (\eta(F^*, c_1) \cup \eta(F^*, c_2))$. Thus, $\alpha(T) \le 3$, a contradiction. Next, suppose that $z = 2$. It follows that $\ell(F^*) = \{2\}$. Hence $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} \ne \bar{F}^*$. Indeed suppose that for some $F_1 \in E(H)$ with $\bar{F}_1 \ne \bar{F}^*$, we have $l(F_1) = \{2\}$. Then consider the cycle $C = \{F^*, F_1, F_2, \}$, where $F_2 \in E(H)$ and $\bar{F}_2 \ne \bar{F}^*$. Now it follows that $3 \in \ell(E(C) \setminus F_1)$, contrary to (4.10). It follows from (6.2) that for every $F \in E(H)$ with $\ell(F) = \{2\}$, $\eta(F) = \eta(F, c_1) \cup \eta(F, c_2)$. Hence, T is the union of two strong cliques $\bigcup\{\eta(F, c_1) \mid F \in E(H)\}$ and $\bigcup\{\eta(F, c_2) \mid F \in E(H)\}$. Thus, $\alpha(T) \le 2$, a contradiction. Therefore, we may assume that $z = 1$. Now T is a strong clique and $\alpha(T) = 1$, a contradiction. This proves (iii). □

(iv) *If $U(H)$ is isomorphic to K_4, then G is resolved.*

Since every edge of H is in a cycle of length four, (4.2) implies that $\ell(F) = \{1\}$ for all $F \in E(H)$. It follows that T, regarded as a graph, is the line graph of K_4. But now, $\alpha(T) \le 2$, a contradiction. This proves (iv). □

(v) *For $t \ge 2$, if $U(H)$ is isomorphic to $K_{2,t}$ or $K_{2,t}^+$, then G is resolved.*

Let Y and Z be such that Y is a stable set and Z satisfies $|Z| = 2$. Write $Y = \{y_1, \dots, y_t\}$ and $Z = \{z_1, z_2\}$. Let $E' \in E(H)$ be the set of edges $F \in E(H)$ with $\bar{F} = \{z_1, z_2\}$. Since every edge in $E(H) \setminus E'$ is in a cycle of length four, (4.2) implies that $\ell(F) = \{1\}$ for all $F \in E(H) \setminus E'$. But now, y_1 and y_2 are nonadjacent clones in H that satisfy the assumptions of (6.1) and therefore G is resolved by (6.1). This proves (v). □

Thus, it follows from (ii)–(v) that G is resolved. This proves (6.3). □

6.3. *Strip-structures that are not 2-connected*

Let T be a connected nonbasic claw-free trigraph and let (T, H, η) be an optimal representation of T. We say that a block B of H is a *leaf-block* if B contains exactly one cut-vertex of H. In Fig. 3, for example, the block labeled $K_{2,4}^+$ is a leaf-block. We call a strip-block that corresponds to a leaf-block in H a *leaf strip-block*.

Let G be an $\mathcal{F}$-free nonbasic claw-free trigraph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H. Consider the strip-block (D, Y) of (H, η) at B. Because B is a leaf-block, there is a unique $y \in Y$. Construct the graph D'

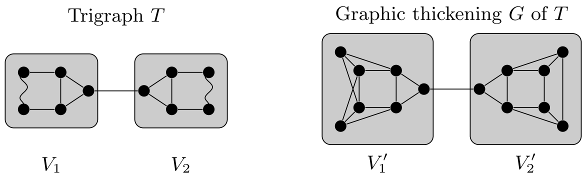


Fig. 5. An example of a trigraph T (left) for which it is not possible to determine from the trigraph alone which leaf strip-block is the ordinary block given by (6.4). The graph on the right shows a $\mathcal{F}$-free graphic thickening of T.

from $G|\bigcup\{X_v \mid v \in V(D)\}$ by adding a new vertex y' that is strongly complete to $Y' = \bigcup\{X_v \mid v \in N_D(y)\}$. Then, G contains D' as an induced subgraph. If D' contains no induced heft with end y', then (D, Y) is said to be *ordinary* (*with respect to* G).

It turns out that if we consider two leaf strip-blocks of an $\mathcal{F}$-free claw-free trigraph T, then at least one them has to be ordinary with respect to a fixed $\mathcal{F}$-free thickening of T (we will prove this in (6.4)). In particular, since the pattern multigraphs of the strip-structures that we are interested in at this point are not 2-connected, there exists at least one ordinary leaf strip-block. Our strategy for concluding that graphs with non-2-connected strip-structures are resolved is to consider such an ordinary leaf strip-block, and find a dominant clique contained in it.

We note that, in the definition of an ordinary leaf strip-block, it is necessary to refer to a specific graphic thickening, because in general the leaf strip-block that is ordinary depends on the graphic thickening. Consider, for example, Fig. 5. The diagram on the left depicts an $\mathcal{F}$-free nonbasic claw-free trigraph T and the diagram on the right shows a graphic thickening G of T, where, for $i = 1, 2$, the vertices in V_i' correspond to the vertices in V_i. With respect to the graphic thickening G, the strip-block corresponding to the set V_2 in T is ordinary and the strip-block corresponding to the set V_1 in T is not ordinary. But by rotating the graphic thickening by 180 degrees, it is clear that with respect to a different graphic thickening, it is possible that the left hand side of the cut edge in T is ordinary. In fact, there are exactly two dominant cliques in G, namely $\{u_1, u_2, u_3\}$ and $\{w_1, w_2\}$, which shows that it is not possible to know where to find a dominant clique from the trigraph alone.

6.3.1. Tools

We need a few lemmas on ordinary leaf strip-blocks.

(6.4). *Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Suppose that B_1, B_2 are two distinct leaf-blocks of H. Then, the strip-block of (H, η) at at least one of B_1, B_2 is ordinary.*

Proof. Suppose that for $i = 1, 2$, the strip-block (D_i, Y_i) of (H, η) at B_i is not ordinary. Let D_i', y_i' be as in the definition of the ordinary strip-block (D_i, Y_i). Because B_i is not ordinary, it follows that D_i has an induced heft H_i with end y_i'. Because G is connected and B_1 and B_2 are leaf-blocks, it follows that there exists an induced path $P = p_1\text{-}p_2\text{-}\cdots\text{-}p_k$, with $k \geq 2$, from a vertex in $N(y_1') \cap V(H_1)$ to a vertex in $N(y_2') \cap V(H_2)$, and $V(P^*)$ is disjoint from $V(D_1') \cup V(D_2')$. It follows from the definition of a strip-structure that p_2 is strongly complete to $N(y_1') \cap V(H_1)$ in G and p_{k-1} is strongly complete to $N(y_2') \cap V(H_2)$ in G. Now, $G|V(H_1) \cup V(H_2) \cup V(P)$ is a skipping rope, a contradiction. This proves (6.4). □

We have the following useful properties of ordinary strip-blocks:

(6.5). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, all of the following hold:*

(a) *D contains no weakly induced heft with its end in Y;*
(b) *D contains no weakly induced cycle of length at least five;*
(c) *B is of the $\mathcal{B}_2$ type;*
(d) *for every cycle C in B, $\ell(E(C)) \subseteq \{3, 4\}$.*

Proof. Part (a) follows immediately from the definition of D, Y, and from (2.1). For part (b), suppose that D contains a weakly induced cycle $C = c_1\text{-}c_2\text{-}\cdots\text{-}c_k\text{-}c_1$ of length $k \geq 5$. Since T is $\mathcal{F}$-free, it follows that $k \in \{5, 7\}$. Since every vertex in Y is simplicial in D, it follows that $Y \cap V(C) = \emptyset$. However, since D is connected, there exists a path P from a vertex $y \in Y$ to a vertex in $V(C)$ with interior in $V(D) \setminus Y$. From the symmetry, we may write $P = p_1\text{-}p_2\text{-}\cdots\text{-}p_m$, where $m \geq 2$ and $p_1 = y$ and $p_m = c_1$. Since P is shortest, it follows that, for $1 \leq j < m - 1$, p_j is anticomplete to $V(C)$. We first claim that p_{m-1} does not have two antiadjacent neighbors $c, c' \in V(C)$. For suppose it does. Since p_1 is a simplicial vertex, it follows that $m \geq 3$. But now, p_{m-1} is complete to the triad $\{c, c', p_{m-2}\}$, a contradiction. Thus, p_{m-1} does not have two antiadjacent neighbors $c, c' \in V(C)$. If p_{m-1} is anticomplete to $\{c_2, c_k\}$, then c_1 is complete to the triad $\{c_2, c_k, p_{m-1}\}$, a contradiction. Thus, since p_{m-1} is not complete to $\{c_2, c_k\}$, we may assume that p_{m-1} is strongly adjacent to c_2 and strongly antiadjacent to c_k. Since every vertex in $V(C) \setminus \{c_1, c_2\}$ is antiadjacent to one of c_1, c_2, it follows that p_{m-1} is strongly anticomplete to $V(C) \setminus \{c_1, c_2\}$. Now, $T|(V(P) \cup V(C))$ is a weakly induced heft with end $y \in Y$, a contradiction. This proves (b). Part (c) follows from part (b), (4.2), and the fact that if B is of the $\mathcal{B}_1$ type, then D contains a weakly induced cycle of length at least five. Part (d) follows immediately from part (b) and (4.7). This proves (6.5). □

This lemma implies that some types of strips $\mathcal{Z}_i$ cannot occur in ordinary blocks.

(6.6). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H such that the strip-block of (H, η) at B is ordinary, and let $F \in E(B)$. Then, the strip of (H, η) at F is not isomorphic to a member of $\mathcal{Z}_4 \cup \mathcal{Z}_7 \cup \mathcal{Z}_8$.*

Proof. Let (J, Z) be the strip of (H, η) at F. For notational convenience, we may assume that (J, Z) is a member of $\mathcal{Z}_4 \cup \mathcal{Z}_7 \cup \mathcal{Z}_8$ (as opposed to isomorphic to a member of that family). We will go through the classes of strips one by one. It follows from (6.5) that J contains no weakly induced cycle of length five. First suppose that $(J, Z) \in \mathcal{Z}_4$. Let $T, a_1, a_2, c_1, b_2, b_1$ be as in the definition of $\mathcal{Z}_4$. Then, a_1-a_2-c_1-b_2-b_1-a_1 is a weakly induced cycle of length five in J, a contradiction. Thus, $(J, Z) \notin \mathcal{Z}_4$. Next, suppose that $(J, Z) \in \mathcal{Z}_7$. Let $H, H', h_1, \ldots, h_5$ be as in the definition of $\mathcal{Z}_7$. Since h_1-h_2-$\cdots$-h_5-h_1 is a cycle of length five in H, it follows that J has an induced cycle of length five, contrary to (6.5). Now, suppose that $(J, Z) \in \mathcal{Z}_8$. Let $A, B, C, X, d_1, d_3, d_4$ be as in the definition of $\mathcal{Z}_8$. Because $A \setminus X$ is not strongly complete to $B \setminus X$, there exist antiadjacent $a \in A$ and $b \in B$. But now, d_1-a-d_3-d_4-b-d_1 is a weakly induced cycle of length five, a contradiction. This proves (6.6). □

The following lemma is a counterpart of (2.10) for ordinary strip-blocks.

(6.7). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf-block of H and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Suppose that (K_1, K_2) is a homogeneous pair of cliques in D such that K_1 is not strongly complete and not strongly anticomplete to K_2, and $V(K_1) \cup V(K_2)$ is strongly anticomplete to Y. For $\{i, j\} = \{1, 2\}$, let $N_i = N(K_i) \setminus N[K_j]$. If the unique vertex $y \in Y$ has a neighbor in both N_1 and N_2, then G is resolved.*

Proof. Let $M = V(T) \setminus (N[K_1] \cup N[K_2])$. For $v \in V(T)$, let X_v denote the corresponding clique in G. Let $K_1' = \bigcup\{X_v \mid v \in K_1\}$ and define K_2', N_1', N_2', Y', M' analogously. Let $Z' = (N(K_1') \cap N(K_2')) \setminus (K_1' \cup K_2')$. Since (K_1, K_2) is a homogeneous pair of cliques, it follows that, for $\{i, j\} = \{1, 2\}$, N_i' is complete to K_i' and anticomplete to K_j', and Z' is complete to $K_1' \cup K_2'$. Hence, from the fact that K_1' is not anticomplete to K_2' and the fact that G is claw-free, it follows that N_1' and N_2' are cliques. Z' is anticomplete to M', because if $z \in Z'$ has a neighbor $u \in M'$, then let $a \in K_1'$, $b \in K_2'$ be nonadjacent and observe that z is complete to the triad $\{a, b, u\}$, contrary to (2.2). Notice that $Y' \subseteq M'$.

From the assumptions of the lemma, it follows that there exist $x_1 \in N_1'$, $x_2 \in N_2'$ and $y \in Y'$ such that y is complete to $\{x_1, x_2\}$. It follows from the fact that the vertex in Y is simplicial in T that x_1 and x_2 are adjacent. We start with the following claim.

(i) *Suppose that there exist $a_1, a_2 \in K_1'$, $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. Then, Z' is complete to N_1'.*

We may assume that $Z' \neq \emptyset$, because otherwise we are done. We first claim that Z' is complete to x_1. For suppose that $z \in Z'$ is nonadjacent to x_1. If z is nonadjacent to x_2, then x_1-a_2-z-b-x_2-x_1 is an induced cycle of length five, a contradiction. Therefore, z is adjacent to x_2. But now, $G|\{y, x_1, a_1, z, x_2, a_2, b\}$ is an induced heft $\mathcal{H}_3(0)$ with end $y \in Y'$, a contradiction. This proves that Z' is complete to x_1.

Now let $p \in N_1'$ and suppose that p is nonadjacent to some $z \in Z'$. Since x_1 is complete to $\{p, y, z\}$, it follows from (2.2) that p is adjacent to y. Since y is a simplicial vertex, and $\{p, x_2\} \in N(y)$, it follows that p is adjacent to x_2. Now, it follows from the previous argument that p is complete to Z', a contradiction. This proves (i). □

We claim that Z' is a clique. For suppose that $z, z' \in Z'$ are nonadjacent. From the fact that K_1' is not complete and not anticomplete to K_2', it follows that either there exist $a_1, a_2 \in K_1'$, $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2, or there exist $a_1, a_2 \in K_2'$, $b \in K_1'$ such that b is adjacent to a_1 and nonadjacent to a_2. Thus, it follows from (i) that Z' is complete to at least one of N_1', N_2'. We may assume from the symmetry that Z' is complete to N_1'. But now x_1 is complete to the triad $\{y, z, z'\}$, contrary to (2.2). Thus, Z' is a clique. The last claim deals with an easy case:

(ii) *If some vertex in K_1' is complete to K_2', then the lemma holds.*

Suppose that $a_1 \in K_1'$ is complete to K_2'. First observe that no vertex in K_1' has both a neighbor and a nonneighbor in K_2', because if $a_2 \in K_1'$ has a neighbor $b_1 \in K_2'$ and a nonneighbor $b_2 \in K_2'$, then $G|\{x, x_1, x_2, a_1, a_2, b_1, b_2\}$ is an induced heft $\mathcal{H}_3(0)$ with end $y \in Y'$, a contradiction. It follows that every vertex in K_1' is either complete or anticomplete to K_2'. Since K_1' is not complete to K_2', it follows that there exists $a_2 \in K_1'$ that is anticomplete to K_2'. Now it follows from (i) that Z' is complete to N_1'. Thus, a_2 is a simplicial vertex and the lemma holds by (2.8). This proves (ii). □

It follows from (ii) and the symmetry that we may assume that, for $\{i, j\} = \{1, 2\}$, no vertex in K_i' is complete to K_j'. Thus, it follows from (i) and the fact that K_1' is not complete and not anticomplete to K_2' that Z' is complete to $N_1' \cup N_2'$. We claim that $K = K_1' \cup Z' \cup N_1'$ is a dominant clique. For suppose not. Then there exists a maximal stable set S in G such that $S \cap K = \emptyset$. Let $a \in K_1'$. Since $N(a) \subseteq K \cup K_2'$, it follows that a has a neighbor in $S \cap K_2'$, because otherwise we may add a to S and obtain a larger stable set. In particular, $S \cap K_2' \neq \emptyset$ and, since K_2' is a clique, $|S \cap K_2'| = 1$. But now, the unique vertex v in $S \cap K_2'$ is complete to K_1', a contradiction. This proves that K is a dominant clique, thus proving (6.7). □

6.3.2. One-edge ordinary leaf-blocks

The most tedious ordinary leaf blocks that we have to deal with are the blocks B that consist of just one edge. In principle, there are 15 different types of strips that we need to deal with. Lemmas (4.5) and (6.6) already ruled out six of them. Lemmas (6.8)–(6.19) deal with the remaining nine types of strips.

(6.8). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_1$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. Write $\bar{F} = \{f_1, f_2\}$. From the symmetry, we may assume that f_1 is a cut-vertex of H. Since J is a linear interval trigraph, we may order the vertices of $V(J \setminus Z)$ as $v_1, \ldots, v_n$ such that for $1 \leq i < k \leq j \leq n$, if v_i is adjacent to v_j, then v_k is strongly adjacent to v_i and v_j. From the symmetry, we may assume that $v_1 \in \eta(F, f_1)$. Now let i be smallest such that v_n is adjacent to v_i. It follows from the definition of $v_1, \ldots, v_n$ that $N(v_n) = \{v_i, v_{i+1}, \ldots, v_{n-1}\}$ and $N(v_n)$ is a strong clique. Therefore, v_n is a simplicial vertex and the result follows from (2.8). This proves (6.8). □

(6.9). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, X) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_2$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_2$ to which (J, Z) is isomorphic. It follows from (6.5) that J contains no weakly induced cycle of length five. Let $A, B, C, X, n, \{a_i\}, \{b_i\}, \{c_i\}$ be as in the definition of $\mathcal{Z}_2$. Let $A' = A \setminus X, B' = B \setminus X, C' = C \setminus X$. Let $\{f_1, f_2\} = \bar{F}$. We may assume that f_1 is a cut-vertex of H and, from the symmetry, that $A' = \eta(F, f_1)$. We first make the following easy observation:

(i) *There are no distinct $i, j, k \in [n]$ such that $a_i, a_j \in A'$, $b_i, b_k \in B'$ and $c_i \in C'$.*

Suppose that such a_i, a_j, b_i, b_k, c_i exist. Then, c_i-a_j-a_i-b_i-b_k-c_i is a weakly induced cycle of length five, a contradiction. □

First suppose that $|B'| = 1$. Let i be such that $b_i \in B'$. Since $N(b_i) = (C' \setminus \{c_i\}) \cup \{a_i\}$, it follows from the definition of $\mathcal{Z}_2$ that b_i is simplicial and hence G is resolved by (2.8). So we may assume that $|B'| \geq 2$.

(ii) *If there exists $i \in [n]$ such that $a_i \in A'$, $b_i \in B'$, $c_i \in C'$, then G is resolved.*

Without loss of generality we may assume that $i = 1$. It follows from the definition of $\mathcal{Z}_2$ that c_i is strongly anticomplete to $\{a_1, b_1\}$, and a_1, b_1 are strongly adjacent. First suppose that there exists $j \in \{2, \ldots, n\}$ such that $a_j \in A'$ and $b_j \in B'$. We may assume that $j = 2$. It follows from i that $A = \{a_1, b_2\}$ and $B = \{b_1, b_2\}$. If a_2 is semiadjacent to b_2, then a_1-a_2-c_1-b_2-b_1-a_1 is a weakly induced cycle of length five, a contradiction. Thus, a_2 is strongly adjacent to b_2.

We claim that $K = \{a_2, b_2\} \cup (C' \setminus \{c_2\})$ is a dominant clique in T. Clearly, K is a strong clique. So suppose that there exists a stable set $S \subseteq V(T)$ that covers K. Since, in particular, S covers c_1. Because $\{a_1, b_1, c_2\}$ is a strong clique, it follows that $S \cap \{a_1, b_1, c_2\} = \{c_2\}$. But now, no vertex in S covers b_2, a contradiction. Thus K is a dominant clique and G is resolved by (2.6).

So we may assume that for no $j \in \{2, \ldots, n\}$, both $a_j \in A'$ and $b_j \in B'$. By this and (i), it follows from the fact that $|B'| \geq 2$ that $A' = \{a_1\}$. Now let $X_1 = (B \setminus \{b_1\}) \cup \{c_1\}$ and $X_2 = (C \setminus \{c_1\}) \cup \{a_1, b_1\}$. Observe that X_1 and X_2 are strong cliques. Since $N(X_2) \subseteq X_1$, it follows from (2.7) that G is resolved. This proves (ii).

In view of (ii), we may assume that there is no $i \in [n]$ such that $a_i \in A'$, $b_i \in B'$, $c_i \in C'$. Now let $B^* = \{b_i : i \in [n], c_i \in C'\}$. It follows that B^* is strongly anticomplete to A' and $B' \setminus B^*$ is strongly complete to C. If $B^* \neq \emptyset$, then B^* is a strong clique, $N(B^*) \subseteq (B' \setminus B^*) \cup C$, and G is resolved by (2.7). So we may assume that $B^* = \emptyset$. Now, $B' \cup C$ is a strong clique and $N(B' \cup C) \subseteq A'$, and G is resolved by (2.7). This proves (6.9). □

(6.10). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_3$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_3$ to which (J, Z) is isomorphic. It follows from (6.5) that J contains no weakly induced cycle of length five. Let $H, h_1, \ldots, h_5, Z$ be as in the definition of $\mathcal{Z}_3$. Write $\bar{F} = \{f_1, f_2\}$. From the symmetry, we may assume that f_1 is a cut-vertex of H and we may assume that $h_2h_3 \in \eta(F, f_1)$.

(i) *If there exists $v \in V(H)$ that is adjacent to h_2 and not to h_3, h_4, then T is resolved.*

Suppose such v exists. It follows from the definition of a line trigraph that h_2v is a simplicial vertex in T and hence T is resolved by (2.8). This proves (i). □

By (i) and the symmetry, we may assume that no vertex is adjacent to h_2 and nonadjacent to h_3, h_4, and that no vertex is adjacent to h_4 and nonadjacent to h_2, h_3. Thus, we may partition $V(H) \setminus \{h_1, h_2, h_3, h_4, h_5\}$ into sets X, Y_2, Y_3, Y_4, W such that X is complete to $\{h_2, h_3, h_4\}$ and, for $i \in \{2, 3, 4\}$, Y_i is anticomplete to h_i and complete to $\{h_2, h_3, h_4\} \setminus \{h_i\}$, and W is complete to h_3 and anticomplete to $\{h_2, h_4\}$.

(ii) *If $X \cup Y_3 \neq \emptyset$, then $W = \emptyset$.*

Suppose there exists $x \in X \cup Y_3$ and $w \in W$. Then, h_2h_3-h_3w-h_3h_4-h_4x-h_3x-h_2h_3 is a weakly induced cycle of length five, a contradiction. This proves (ii). □

(iii) *If $X \neq \emptyset$, then G is resolved.*

Suppose that $X \neq \emptyset$. It follows from (ii) that $W = \emptyset$. Let $x \in X$. If there exists $y_2 \in Y_2$, then y_2-h_3-h_4-x-h_2-y_2 is a cycle of length five, and thus, by the definition of a line trigraph, J contains a weakly induced cycle of length five, a contradiction. If there exists $y_3 \in Y_3$, then y_3-h_2-x-h_3-h_4-y_3 is a cycle of length five, a contradiction. If there exists $x' \in X$, $x' \neq x$, then h_2-x-h_3-x'-h_4-h_2 is a cycle of length five in H, a contradiction. From this and the symmetry, it follows that $Y_2 = Y_3 = Y_4 = \emptyset$ and $|X| = 1$. Now let $A = \{h_3h_4, h_4x\}$ and let $B = \{h_2h_3, h_2x, h_3x\}$. Now, A and B are strong cliques and $N(A) = B$. Therefore, G is resolved by (2.7). This proves (iii). □

It follows from (iii) that we may assume that $X = \emptyset$. Now first suppose that $Y_3 \neq \emptyset$. It follows from (ii) that $W = \emptyset$. If there exists $y_4 \in Y_4$, then h_2-y_4-h_3-h_4-y_3-h_2 is a cycle of length five, and hence T contains a weakly induced cycle of length five, a contradiction. Therefore, by the symmetry, $Y_2 = Y_4 = \emptyset$. Let $A = \{h_4y_3 \mid y_3 \in Y_3\} \cup \{h_3h_4\}$ and let $B = \{h_2y_3 \mid y_3 \in Y_3\} \cup \{h_2h_3\}$. Then, $N(A) \subseteq N(B)$ and A and B are strong cliques and, thus, G is resolved by (2.7). Thus, we may assume that $Y_3 = \emptyset$. Now, let $A = \{h_4y_2 \mid y_2 \in Y_2\} \cup \{h_3h_4\}$ and let $B = \{h_3y \mid y \in Y_2 \cup Y_4 \cup W\} \cup \{h_2h_3\}$. Then, $N(A) \subseteq N(B)$ and A and B are strong cliques and, thus, G is resolved by (2.7). This proves (6.10). □

For the next type of strip, we need a lemma from [3]. Before we can state the result, we need some definitions. Let $\bar{C}_7$ be the complement graph of a 7-cycle. We say that a trigraph T is *of the $\bar{C}_7$ type* if $V(T)$ can be partitioned into seven nonempty strong cliques $W_1, \ldots, W_7$ such that for all $i \in [7]$, (subscript arithmetic is modulo 7)

(1) W_i is strongly complete to W_{i+1};
(2) W_i is complete to W_{i+2};
(3) W_i is strongly anticomplete to W_{i+3}.

For a trigraph T, let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 \subseteq V(T)$ be strong cliques such that, for $i = 1, 2, 3, 4$, (with subscript arithmetic modulo 4)

(1) if $i \in \{1, 3\}$, then A_i is complete to A_{i+1}, and if $i \in \{2, 4\}$, then A_i and A_{i+1} are linked, and
(2) A_i is strongly anticomplete to A_{i+2}, and
(3) B_i is strongly complete to $A_i \cup A_{i+1}$ and strongly anticomplete to $A_{i+2} \cup A_{i+3}$, and
(4) B_i is strongly anticomplete to B_j for $i \neq j$, and
(5) if $B_i \neq \emptyset$, then A_i is complete to A_{i+1}, and
(6) no vertex in A_i has antineighbors in both A_{i-1} and A_{i+1}.

We call such $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ a *C_4-structure* in T. If, for T, there exists a C_4-structure $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ such that $V(T) = A_1 \cup \cdots \cup A_4 \cup B_1 \cup \cdots \cup B_4$, then we say that *$T$ admits a C_4-structure*. The following lemma states that there are three types of long circular interval trigraphs that have no semiholes of length at least five:

(6.11) *((5.7) in [3]). Let T be a long circular interval graph that has no semihole of length at least five. Then, either*

(i) *T is a linear interval trigraph, or*
(ii) *T is of the $\bar{C}_7$ type, or*
(iii) *T admits a C_4-structure.*

This allows use to deal with strips that are circular interval trigraphs.

(6.12). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_6$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from the definition of $\mathcal{Z}_6$ that J is a long circular interval graph that contains a simplicial vertex $z \in Z$. We may assume that J is not a linear interval trigraph, because then the result follows from (6.8). It follows from (6.5) and the fact that G is $\mathcal{F}$-free that J contains no weakly induced cycle of length at least five, and in particular, J contains no semihole of length at least five. We may assume that G is not resolved.

(i) *J admits a C_4-structure $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ and there exists $i \in [4]$, such that $z \in B_i$ and $B_j = \emptyset$ for $j \in [4] \setminus \{i\}$.*

Since a trigraph of the $\bar{C}_7$ type contains no simplicial vertex, it follows from (6.11) that J admits a C_4-structure $(A_1, \ldots, A_4, B_1, \ldots, B_4)$. It follows from properties (1) and (2) of a C_4-structure that no vertex in $A_1 \cup A_2 \cup A_3 \cup A_4$ is simplicial, and hence that $z \in B_i$ for some $i \in [4]$. It follows from property (5) that A_i is strongly complete to A_{i+1}. Now suppose that $B_j \neq \emptyset$ for some $j \in [4] \setminus \{i\}$. By properties (3)–(5), every vertex in B_j is simplicial in T and therefore G is resolved by (2.8), a contradiction. It follows that $B_j = \emptyset$ for all $j \in [4] \setminus \{i\}$. This proves (i). □

Let $A_1, \ldots, A_4, i$ be as in the statement of (i).

(ii) *A_i is strongly complete to A_{i+3} and A_{i+2} is strongly complete to A_{i+1}.*

We first claim that there exist antiadjacent $a_{i+2} \in A_{i+2}$ and $a_{i+3} \in A_{i+3}$. For suppose not. Then, $A_{i+2} \cup A_{i+3}$ and $A_i \cup A_{i+1}$ are strong cliques and $N(A_{i+2} \cup A_{i+3}) \subseteq A_i \cup A_{i+1}$. Therefore, G is resolved by (2.7), a contradiction. This proves the claim.

It follows from property (6) that a_{i+2} is strongly complete to A_{i+1}, and a_{i+3} is strongly complete to A_i. Now suppose that there exist antiadjacent $a_i \in A_i$ and $a'_{i+3} \in A_{i+3}$. Then, by property (6), a'_{i+3} is strongly complete to A_{i+2}. Now, a_i-a_{i+1}-a_{i+2}-a'_{i+3}-a_{i+3}-a_i, with $a_i \in A_i$, is a weakly induced cycle of length five, a contradiction. This proves that A_i is strongly complete to A_{i+3} and therefore, by the symmetry, that A_{i+2} is strongly complete to A_{i+1}, completing the proof of (ii). □

It follows from (ii) that (A_{i+2}, A_{i+3}) is a homogeneous pair of cliques that satisfies the assumptions of (6.7), and therefore G is resolved by (6.7). This proves (6.12). □

(6.13). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_9$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (6.5) that J contains no weakly induced cycle of length five. We may assume that G is not resolved. For convenience, we identify the vertices of J with the vertices of the member of $\mathcal{Z}_9$ to which (J, Z) is isomorphic.

Let $A, B, C, D, \{a_i\}, \{b_i\}$, and n be as in the definition of $\mathcal{Z}_9$. Recall that, every vertex $d \in D$ is strongly adjacent to one of $a_i, b_i, i \in [n]$, and strongly antiadjacent to the other. For $i \in [n]$, we say that two vertices $d_1, d_2 \in D$ *agree* on a_ib_i if $\{d_1, d_2\}$ is strongly complete to one of a_i, b_i, and strongly anticomplete to the other. They *disagree* on a_ib_i otherwise.

(i) *If $d_1, d_2 \in D$ disagree on a_ib_i for some $i \in [n]$, then d_1, d_2 disagree on a_jb_j for every $j \in [n]$.*

From the symmetry, we may assume that d_1, d_2 disagree on a_1b_1 and d_1, d_2 agree on a_2b_2. From the symmetry, we may also assume that d_1 is strongly complete to $\{a_1, a_2\}$ and strongly anticomplete to $\{b_1, b_2\}$, and d_2 is strongly complete to $\{b_1, a_2\}$ and strongly anticomplete to $\{a_1, b_2\}$. But now, d_1-a_1-b_2-b_1-d_2-d_1 is a weakly induced cycle of length five, a contradiction. This proves (i). □

It follows from (i) that D may be partitioned into two sets D_1, D_2, such that, for $i = 1, 2$, the vertices in D_i agree on all pairs $a_jb_j, j \in [n]$, and whenever $d_1 \in D_1$ and $d_2 \in D_2$, then d_1, d_2 disagree on all pairs $a_jb_j, j \in [n]$. For $\{i, j\} = \{1, 2\}$, let $A_i \subseteq A$, $B_i \subseteq B$ be the vertices in A, B, respectively, that are strongly complete to D_i and strongly anticomplete to D_j. It follows that $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $D = D_1 \cup D_2$ and, for $i = 1, 2$, $A_i \cup B_i \cup D_i$ is a strong clique.

(ii) *A_1, A_2, B_1, B_2 are all nonempty, A_1 is strongly anticomplete to B_2, and A_2 is strongly anticomplete to B_1.*

Since, for $\{i, j\} = \{1, 2\}$, every vertex in A_i has an antineighbor in B_j and vice versa, it follows that A_i, B_j are either both empty or both nonempty. If $A_1 \cup B_2 = \emptyset$, then $C \cup A_2 \cup B_1$ is a strong clique, $N(C \cup A_2 \cup B_1) \subseteq D$, and D is a strong clique, and thus G is resolved by (2.7), a contradiction. Hence, by the symmetry, A_1, A_2, B_1, B_2 are all nonempty.

Since every vertex in A_1 has an antineighbor in B, it follows that A_1 is not strongly complete to B_2. Now observe that (A_1, B_2) is a homogeneous pair of cliques that satisfies the assumptions of (6.7). It follows from (6.7) and the fact that G is not resolved that A_1 is strongly anticomplete to B_2. Symmetrically, it follows that A_2 is strongly anticomplete to B_1, thus proving (ii). □

Now suppose for a contradiction that $D_1 = \emptyset$. It follows that $X_1 = A_1 \cup B_1 \cup C$ is a strong clique and, since $D_1 = \emptyset$, $N(X_1) \subseteq A_2 \cup B_2 \cup D_2$, which is also a strong clique. Therefore, it follows from (2.7) that G is resolved. It follows from the symmetry that D_1 and D_2 are both nonempty.

(iii) *C is strongly complete to at least one of D_1, D_2.*

If $c \in C$ has antineighbors $d_1 \in D_1, d_2 \in D_2$, then d_1-a_1-c-b_2-d_2-d_1 with $a_1 \in A_1, b_2 \in B_2$ is a weakly induced cycle of length five, a contradiction. We may assume that some $c_1 \in C$ has an antineighbor $d_1 \in D_1$, and some $c_2 \in C$ has an antineighbor $d_2 \in D_2$. By the previous argument, $c_1 \neq c_2$, c_1 is strongly adjacent to d_2, and c_2 is strongly adjacent to d_1. Let $a \in A_1$ and $b \in B_2$. Then, $J|\{y, d_1, c_2, c_1, d_2, a, b\}$ contains a weakly induced heft $\mathcal{H}_3(0)$ with end $y \in Y$, contrary to (6.5). This proves (iii). □

So we may assume that C is strongly complete to D_1. Now, let $K = A_1 \cup B_1 \cup D_1 \cup C$. We claim that K is a dominant clique in T. For suppose that there exists a stable set $S \subseteq V(T)$ that covers K. Since, in particular, S covers B_1, it follows that $S \cap A_2 \neq \emptyset$. Since $A_2 \cup B_2 \cup D_2$ is a strong clique, it follows that $|S \cap (A_2 \cup B_2 \cup D_2)| = 1$. But this implies that S does not cover A_1, a contradiction. Thus, K is a dominant clique in T and G is resolved by (2.6). This proves (6.13). □

In the remaining cases, we will always deal with strips that are hex-expansions of three-cliqued strips. We first prove a useful lemma on hex-expansions:

(6.14). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$. Suppose that the strip block of (H, η) at B is ordinary and that the strip of (H, η) at F is a trigraph that is a hex-expansion of a three-cliqued trigraph (T', A, B, C). Let V_1, V_2, V_3 be as in the definition of the hex-expansion. Then, either:*

(a) *G is resolved, or*
(b) *B and C are nonempty, and $V_1 \cup V_2 \cup V_3$ is a strong clique.*

Proof. Let (J, Z) be the strip of (H, η) at F. Let (T', A, B, C) be such that J is a hex-expansion of (T', A, B, C) with $z \in A$, and let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (6.5) that J contains no weakly induced cycle of length five. We may assume that G is not resolved, because otherwise outcome (a) holds.

First suppose that $V_1 \cup V_2 \cup V_3$ is a strong clique. If B is empty, then $V_1 \cup C \cup V_2$ is a strong clique that only has neighbors in the strong clique $A \cup V_2 \cup V_3$, and hence G is resolved by (2.7). Thus, by the symmetry, B and C are both nonempty, and hence outcome (b) holds. So we may assume that $V_1 \cup V_2 \cup V_3$ is not a strong clique.

Next, if B is strongly complete to C, then $B \cup C \cup V_1$ is a strong clique that only has neighbors in the strong clique $A_1 \cup V_2 \cup V_3$, and hence G is resolved by (2.7), a contradiction. It follows that B is not strongly complete to C,

(i) *V_1 is strongly complete to one of V_2, V_3.*

Since B is not strongly complete to C, there exist antiadjacent $b \in B$, $c \in C$. First suppose that $v_1 \in V_1$ has antineighbors $v_2 \in V_2$ and $v_3 \in V_3$. Then, v_2-c-v_1-b-v_3-v_2 is a weakly induced cycle of length five, a contradiction. This proves that no vertex in V_1 has an antineighbor in both V_2 and V_3. So we may assume that there exist antiadjacent $v_1 \in V_1$, $v_2 \in V_2$ and antiadjacent $v_1' \in V_2$ and $v_3 \in V_3$. It follows that v_1 is strongly adjacent to v_3 and v_1' is strongly adjacent to v_2. Now, $J|\{z, v_1, v_1', v_2, v_3, b, c\}$ contains a weakly induced heft $\mathcal{H}_3(0)$ with end $z \in Z$, a contradiction. This proves (i). □

In view of (i) and the symmetry, we may assume that V_1 is strongly complete to V_2 and V_1 is not strongly complete to V_3. Let $C' \subseteq C$ be all vertices in C that have a neighbor in A.

(ii) *C' is strongly complete to B.*

Suppose that $c \in C'$ has an antineighbor $b \in B$. Since $c \in C'$, c has a neighbor $a \in A$. Because a is not complete to the triad $\{b, c, z\}$, it follows that a is strongly antiadjacent to b. Now, v_3-a-c-v_1-b-v_3, with $v_1 \in V_1$ and $v_3 \in V_3$ antiadjacent, is a weakly induced cycle of length five, a contradiction. This proves (ii). □

Since B is not strongly complete to C, it follows that $C \setminus C' \neq \emptyset$. If $V_2 = \emptyset$, then $C \setminus C'$ is a strong clique, $N(C \setminus C') \subseteq B \cup C' \cup V_1$ is a strong clique, and thus G is resolved by (2.7), a contradiction. Therefore, $V_2 \neq \emptyset$.

(iii) *There are no $a \in A$, $b, b' \in B$, $c \in C$ such that both a and c are mixed on $\{b, b'\}$.*

Suppose that such a, b, b', c' exist. From the symmetry, we may assume that a is adjacent to b and antiadjacent to b'. Because C' is strongly complete to B, it follows that $c \in C \setminus C'$. Thus, c is strongly antiadjacent to a. If c is adjacent to b and antiadjacent to c', then b is complete to the triad $\{a, b', c\}$, contrary to (2.2). Thus, c is adjacent to b' and antiadjacent to b. Let $v_2 \in V_2$. Now, v_2-a-b-b'-c-v_2 is a weakly induced cycle of length five, a contradiction. This proves (iii). □

Since B is not strongly complete to C, it follows from (ii) that some $b \in B$ and $c \in C \setminus C'$ are antiadjacent. If b and c are semiadjacent, then G is resolved by (6.7) applied to b-c-v_2-v_1-b with $v_1 \in V_1$ and $v_2 \in V_2$, a contradiction. Thus, $c \in C$ is not semiadjacent to any vertex in B. If c is strongly anticomplete to B, then c is simplicial and thus G is resolved by (2.8), a contradiction. Hence, c has a strong neighbor $b' \in B$. Therefore, c is mixed on $\{b, b'\}$ and hence, by (iii), no vertex in A is mixed on $\{b, b'\}$. Since every vertex in $B \setminus \{b, b'\}$ is adjacent to one of b, b', it follows that no vertex in A is mixed on B. Now, (B, C) is a homogeneous pair of cliques that satisfies (6.7) and hence G is resolved by (6.7), a contradiction. This proves (6.14). □

(6.15). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{10}$, then G is resolved.*

Proof. Let $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, c_1, c_2, A, B, C, X$ be as in the definition of $\mathcal{Z}_{10}$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (6.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique. We first note that if $\{b_2, b_3\} \subseteq X$, then $N(c_1) = V_1 \cup V_2 \cup \{c_2\}$, and hence c_1 is a simplicial vertex. Therefore, by (2.8), we may assume that at least one of b_2, b_3 is not in X. It follows from the fact that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$, that $a_2 \in X$. If $d \in X$, then it follows that $N(b_0) = \{b_1, b_2, b_3\} \cup V_1 \cup V_3$ and hence b_0 is a simplicial vertex. Thus, by (2.8), we may assume that $d \notin X$. Now, if $b_2 \notin X$, then b_0-d-a_1-c_2-c_1-b_2-b_0 is a weakly induced cycle of length six, a contradiction. Therefore, $b_2 \in X$ and $b_3 \notin X$. Now, b_0-d-a_1-c_2-c_1-b_3-b_0 is a weakly induced cycle of length six, a contradiction. This proves (6.15). □

(6.16). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{11}$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (6.5) that J contains no weakly induced cycle of length five. Let a_0, b_0, A, B, C, X be as in the definition of $\mathcal{Z}_{11}$ Let $A' = A \setminus (X \cup \{a_0\})$, $B' = B \setminus (X \cup \{b_0\})$, $C' = C \setminus X$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.*, V_1 is strongly complete to $B' \cup C'$, V_2 is strongly complete to $A' \cup C'$, and V_3 is strongly complete to $A' \cup B'$.

It follows from (6.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique, and B' is nonempty. If $a_0 \in X$ or a_0 is strongly antiadjacent to b_0, then $N(b_0) = B' \cup V_1 \cup V_3$ and hence b_0 is a simplicial vertex, and G is resolved by (2.8). So we may assume that $a_0 \notin X$ and a_0 is semiadjacent to b_0.

We claim that $N(C)$ is a strong clique. For suppose not. Then there exist antiadjacent $u_1, u_2 \in N(C)$. Since $N(C) \subseteq A' \cup B' \cup V_1 \cup V_2$, B' is strongly complete to V_1, and A' is strongly complete to V_2, we may assume that $u_1 \in A' \cup V_2$ and $u_2 \in B' \cup V_1$. Because $u_1, u_2 \in N(C)$, there exists a weakly induced path P from u_1 to v_2 such that $V(P^*) \subseteq C$ and $|V(P)| \in \{3, 4\}$. Now, a_0-u_1-P-u_2-b_0-a_0 is a weakly induced cycle of length five or six, a contradiction. □

Before we prove the next lemma, we need a definition and a corresponding result from [3]. Let T be a long circular interval trigraph, and let Σ be a circle with $V(T) \subseteq \Sigma$, and $F_1, \ldots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(T)$ with $|X| = 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(T)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(T) \subseteq L_1 \cup L_2 \cup L_3$. Then $(T, V(T) \cap L_1, V(T) \cap L_2, V(T) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

(6.17) *((5.16) in [3]). Every $(T, L_1, L_2, L_3) \in \mathcal{TC}_2$ is either a linear interval trigraph or contains a semihole of length at least five.*

(6.18). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{13}$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. Let z be the unique vertex in Z. Let T', L_1, L_2, L_3 be as in the definition of $\mathcal{Z}_{13}$. LetV_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.,* V_1 is strongly complete to $L_2 \cup L_3$, V_2 is strongly complete to $L_1 \cup L_3$, and V_3 is strongly complete to $L_1 \cup L_2$. It follows from (6.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique. Therefore, from the symmetry, we may assume that $z \in L_1$. Notice that $(T', V(T') \cap L_1, V(T') \cap L_2, V(T') \cap L_3)$ is a three-cliqued claw-free trigraph that belongs to the class $\mathcal{TC}_2$. It follows from (6.17) that either T' contains a semihole of length at least five, or T' is a linear interval trigraph. Suppose first that T' contains a semihole of length at least five. Then, since T' is an induced subtrigraph of J, it follows that J contains a semihole of length at least five, contrary to (6.5). This proves that T' is a linear interval trigraph. It follows that at least one of the pairs (L_1, L_2), (L_1, L_3), (L_2, L_3) is strongly anticomplete.

First, assume that L_2 is strongly anticomplete to L_3. We may assume that $V_1 \neq \emptyset$, because otherwise L_2 and $L_1 \cup V_3$ are nonempty strong cliques and $N(L_2) \subseteq L_1 \cup V_3$ and G is resolved by (2.7). We first claim that there do not exist $u, w \in L_1$ such that u is complete to L_2 and w is complete to L_3. For suppose such u, w do exist. Then, since every vertex in $L_1 \cup L_2 \cup L_3$ is in a triad, there exists $u' \in L_2$ such that u and u' are semiadjacent and there exists $w' \in L_3$ such that w and w' are semiadjacent. Since every vertex is semiadjacent to at most one other vertex, it follows that $u \neq w$. If u is antiadjacent to w' and w is antiadjacent to u', then v_1-u'-u-w-w'-v_1 with $v_1 \in V_1$ is a weakly induced cycle of length five, contrary to (6.5). If u is adjacent to w' and w is adjacent to u', then v_1-u'-w-u-w'-v_1 with $v_1 \in V_1$ is a weakly induced cycle of length five, contrary to (6.5). Thus, from the symmetry, we may assume that u is strongly antiadjacent to w' and w is strongly adjacent to u'. But now, w is complete to the triad $\{u, u', w'\}$, contrary to (2.2). This proves that there do not exist $u, w \in L_1$ such that u is complete to L_2 and w is complete to L_3. Thus, from the symmetry, we may assume that no vertex in L_1 is complete to L_2. We now claim that $K = V_1 \cup V_3 \cup L_2$ is a dominant clique. For suppose not. Then there exists a stable set $S \subseteq V(T)$ such that S covers K. Since $L_3 \cup V_2$ is strongly anticomplete to L_2, it follows that there exists $s \in L_1$ that is complete to L_2, a contradiction. Thus, K is a dominant clique and G is resolved.

So we may assume that L_2 is not anticomplete to L_3. From the symmetry, we may assume that L_1 is strongly anticomplete to L_2. Moreover, if some $l_2 \in L_2$ and $l_3 \in L_3$ are semiadjacent, then G is resolved by (6.7) applied to the homogeneous pair of cliques $(\{l_2\}, \{l_3\})$ and the lemma holds. So we may assume that there are no semiadjacencies between L_2 and L_3. We claim that $K = V_1 \cup V_3 \cup L_2$ is a dominant clique. For suppose for a contradiction that there exists a stable set S that covers K. Since, in particular, S covers L_2, it follows that there exists $x \in S \cap L_3$ such that x is complete to L_2. Since no vertex in L_3 is semiadjacent to a vertex in L_2, it follows that x is strongly complete to L_2. But this contradicts the fact that x lies in a triad in $L_1 \cup L_2 \cup L_3$. This proves (6.18). □

(6.19). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H with $E(B) = \{F\}$ and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. If the strip of (H, η) at F is isomorphic to a member of $\mathcal{Z}_{15}$, then G is resolved.*

Proof. Let (J, Z) be the strip of (H, η) at F. It follows from (6.5) that J contains no weakly induced cycle of length five. Let $A, B, C, X, v_1, \ldots, v_8$ be as in the definition of $\mathcal{Z}_{15}$. Let V_1, V_2, V_3 be as in the definition of the hex-expansion, *i.e.,* V_1 is strongly complete to $B \cup C$, V_2 is strongly complete to $A \cup C$, and V_3 is strongly complete to $A \cup B$. It follows from (6.14) that we may assume that $V_1 \cup V_2 \cup V_3$ is a strong clique. If v_2 is semiadjacent to v_5, then $(\{v_2\}, \{v_5\})$ form a homogeneous pair of cliques in T that satisfy the assumptions of (6.7) and thus G is resolved by (6.7). Therefore, we may assume that v_2 is strongly antiadjacent to v_5. Moreover, if $X = \emptyset$, then $J|\{v_1, v_2, \ldots, v_8\}$ contains a weakly induced heft $\mathcal{H}_3(1)$, a contradiction. From the symmetry, we may assume that $v_4 \in X$. But now, $N(v_2) \subseteq \{v_1, v_3\} \cup V_2 \cup V_3$ is a strong clique. Thus, v_2 is a simplicial vertex and, hence, G is resolved by (2.8). This proves (6.19). □

6.3.3. Multi-edge ordinary leaf-blocks

The previous subsection dealt with ordinary leaf-blocks that consist of exactly one edge. The following lemmas deal with the remaining cases when an ordinary leaf-block consists of multiple edges. Recall from (6.5) that such a leaf-block B is of the $\mathcal{B}_2$ type, and hence $U(B)$ is isomorphic to one of $K_2, K_3, K_4, K_{2,t}$, or $K^+_{2,t}$ $(t \geq 2)$. We start with the case K_2:

(6.20). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H such that $U(B)$ is isomorphic to K_2, and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, G is resolved.*

Proof. We may assume that G is not resolved. Let $\{u, v\} = V(B)$ such that u is the unique cut-vertex of H that belongs to $V(B)$. It follows from (6.5) that either $\ell(F) \in \{\{1\}, \{2\}\}$ for all $F \in E(B)$, or there exists $F^* \in E(B)$ with $\ell(F^*) \subseteq \{2, 3\}$ and $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$.

First suppose that $\ell(F) \in \{\{1\}, \{2\}\}$ for all $F \in E(B)$. Since G is not resolved, it follows from (6.2) that $\eta(F) = \eta(F, u) \cup \eta(F, v)$ for all $F \in E(H)$ with $\ell(F) = \{2\}$. But now let

$$M_1 = \bigcup\{\eta(F, u) \mid F \in E(H)\} \quad \text{and} \quad M_2 = \bigcup\{\eta(F, v) \mid F \in E(H), \ell(F) = \{2\}\}.$$

It follows from the definition of a strip-structure that M_1 and M_2 are strong cliques and $N(M_2) \subseteq N(M_1)$. Hence, the lemma holds by (2.7).

So we may assume that there exists $F^* \in E(B)$ with $3 \in \ell(F^*)$ and $\ell(F) = \{1\}$ for all $F \in E(H) \setminus \{F^*\}$. It follows from (4.9) that $\ell(F^*) \neq \{3\}$ and hence $\ell(F^*) = \{2, 3\}$. Let $A = \eta(F^*, u)$, $B = \eta(F^*, v)$, and $C = \eta(F^*) \setminus (\eta(F^*, u) \cup \eta(F^*, v))$. It follows from (4.12) that C is a strong clique. Let (J, Z) be the strip of (H, η) at F^*, let z_1 be the unique vertex in Z that is strongly complete to A and let z_2 be the unique vertex in Z that is strongly complete to B. Let $M = \bigcup\{\eta(F) \mid F \in E(H) \setminus \{F^*\}\}$. It follows that M is strongly complete to $A \cup B$ and strongly anticomplete to C.

(i) *At least one of A, B is not mixed on C.*

Suppose that A and B are both mixed on C. Construct the trigraph T' from T by making A strongly anticomplete to B. It follows from (4.8) applied to T' that there exists a weakly induced path $P = p_1\text{-}p_2\text{-}p_3\text{-}p_4$ in T' with $p_1 \in A$, $p_2, p_3 \in C$, and $p_4 \in B$. If p_1, p_4 are antiadjacent in T, then $x\text{-}P\text{-}x$ with $x \in M$ is a weakly induced cycle of length five in D, a contradiction. Thus, p_1, p_4 are adjacent. But now, p_1 is complete to the triad $\{z_1, p_2, p_4\}$, contrary to (2.2). This proves (i). □

Now let $A' \subseteq A$, $B' \subseteq B$ be the vertices in A, B, respectively, that have a neighbor in C. It follows that $A \setminus A'$ is strongly anticomplete to C and, because J is claw-free, to B'. It follows that $B \setminus B'$ is strongly anticomplete to C and, because J is claw-free, to A'.

(ii) *$A = A'$ and $B = B'$.*

If $N(C)$ is a strong clique, then it follows from (2.7) that G is resolved, a contradiction. Thus, there exist $c_A, c_B \in C$ such that c_A has a neighbor $a \in A'$, c_B has a neighbor $b \in B'$, and c_A and c_B are antiadjacent. If we cannot choose $c_A = c_B$, then the path $a\text{-}c_A\text{-}c_B\text{-}b$ is a weakly induced path of length 4 and hence $4 \in \ell(F^*)$, a contradiction. Thus, we may assume that $c_A = c_B$. We claim that $A \setminus A'$ is strongly anticomplete to $B \setminus B'$. For suppose that there exist adjacent $a \in A \setminus A'$ and $b \in B \setminus B'$. Then, $a\text{-}a'\text{-}c_A\text{-}b'\text{-}b\text{-}a$ is a weakly induced cycle of length five, a contradiction. Now suppose that one of $A \setminus A'$, $B \setminus B'$ is nonempty. Then, because $N[A \setminus A'] \subseteq \bigcup\{\eta(F, u) \mid F \in E(H), u \in \bar{F}\}$ and $N[B \setminus B'] = M$ are strong cliques, it follows from (2.7) that G is resolved, a contradiction. Therefore, $A = A'$ and $B = B'$. □

By (i), at most one of A, B is mixed on C. Since every vertex in $A \cup B$ has a neighbor in C, it follows that at least one of A, B is strongly complete to C. If B is strongly complete to C, then, because $N[B \cup C] \subseteq M \cup A$, (2.7) implies that G is resolved. Thus, we may assume that A is strongly complete to C and B is not strongly complete to C. Let $B'' \subseteq B$ be the set of vertices in B that are not strongly complete to C. It follows from our assumptions that $B'' \neq \emptyset$. Since J is claw-free, it follows that B'' is strongly anticomplete to A. Now, (B'', C) is a homogeneous pair of cliques that satisfies the assumptions of (6.7) and, thus, G is resolved by (6.7). This proves (2.8). □

This leaves the cases K_3, K_4, $K_{2,t}$ and $K_{2,t}^+$, all of which we deal with in the next lemma:

(6.21). *Let G be an $\mathcal{F}$-free nonbasic claw-free graph and let (T, H, η) be an optimal representation of G. Let B be a leaf block of H such that $U(B)$ is isomorphic to K_3, K_4, $K_{2,t}$, or $K_{2,t}^+$ for some $t \geq 2$, and suppose that the strip-block (D, Y) of (H, η) at B is ordinary. Then, G is resolved.*

Proof. Let $V(B) = \{v_1, \ldots, v_k\}$ with $k = |V(B)|$. We may assume that v_1 is the unique cut vertex of H in $V(B)$.

(i) *If $U(B)$ is isomorphic to K_3, then G is resolved.*

From (6.5), it follows that $z \leq 2$ for all $z \in \ell(F)$ with $F \in E(B)$. First suppose that $\ell(F) = \{1\}$ for all $F \in E(H)$ with $\bar{F} = \{v_2, v_3\}$. Then,

$$\bigcup\{\eta(F) \mid F \in E(H), \bar{F} = \{v_2, v_3\}\}$$

is a strong clique and all its neighbors are in the strong clique

$$\bigcup\{\eta(F, v_1) \mid F \in E(H), v_1 \in \bar{F}\}.$$

Thus, the lemma holds by (2.7). So we may assume that there exists $F^* \in E(H)$ with $\bar{F}^* = \{v_2, v_3\}$ and $\ell(F^*) = \{2\}$. We may also assume that G is not resolved. It follows from (6.2) that $\eta(F^*) = \eta(F^*, v_2) \cup \eta(F^*, v_3)$. But now, $(\eta(F^*, v_2), \eta(F^*, v_3))$ is a homogeneous pair of cliques that satisfies the assumptions of (6.7), and thus G is resolved by (6.7). □

(ii) *If $U(B)$ is isomorphic to K_4, then G is resolved.*

Because every edge in $E(B)$ is in a cycle of length four in B, it follows from (4.2) that $\ell(F) = \{1\}$ for all $F \in E(H)$. Now,

$$\bigcup\{\eta(F) \mid F \in E(H), \bar{F} \subseteq \{v_2, v_3, v_4\}\}$$

is a strong clique and all its neighbors are in the strong clique

$$\bigcup\{\eta(F, v_1) \mid F \in E(H), v_1 \in \bar{F}\}.$$

Thus, G is resolved by (2.7). □

(iii) *If $U(B)$ is isomorphic to $K_{2,t}$ or $K^+_{2,t}$ for some $t \geq 2$, then G is resolved.*

Observe that every edge in $E(B)$ is in a cycle of length four in B. Therefore, it follows from (4.2) that $\ell(F) = \{1\}$ for all $F \in E(H)$. Let $V(B) = X \cup Y$ such that X is a stable set of size t and $|Y| = 2$. Let v be the unique cut vertex of H in B. If $v \in Y$, then $p, p' \in X$ satisfy the assumptions of (6.1), and hence that G is resolved. Thus, we may assume that $v \in X$. First assume that $U(B)$ is not isomorphic to $K^+_{2,2}$. If $U(B)$ is isomorphic to $K_{2,t}$, then let $p, p' \in Y$. Otherwise, $t \geq 3$, and let $p, p' \in X \setminus \{v\}$. Now, p and p' satisfy the assumptions of (6.1), and hence G is resolved.

So we may assume that $U(B)$ is isomorphic to $K^+_{2,2}$. We may also assume that G is not resolved. Let $Y = \{y_1, y_2\}$. It follows from (6.5) that $\ell(F) \subseteq \{1, 2\}$ for every $F \in E(H)$ with $\bar{F} = \{y_1, y_2\}$. Moreover, it follows from (4.12) that $\eta(F) = \eta(F, y_1) \cup \eta(F, y_2)$ for every $F \in E(H)$ with $\bar{F} = \{y_1, y_2\}$ and $\ell(F) = \{2\}$. Now, let

$$Z_1 = \bigcup\{\eta(F, y_1) \mid F \in E(H), y_1 \in F\}$$

and

$$Z_2 = \bigcup\{\eta(F, y_2) \mid F \in E(H), y_2 \in \bar{F}, \ell(F) = \{2\}\}.$$

It follows that Z_1 and Z_2 are strong cliques and $N(Z_2) \subseteq N(Z_1)$. Thus, G is resolved by (2.7).

This proves (6.21). □

6.4. Proof of Theorem 1.2

We are finally in a position to prove Theorem 1.2:

Theorem 1.2. *Every connected $\mathcal{F}$-free nonbasic claw-free graph is resolved.*

Proof. Let G be a connected $\mathcal{F}$-free nonbasic claw-free graph. It follows from (2.3) that G is a graphic thickening of some claw-free trigraph that admits a proper strip-structure. Therefore, by (4.2), G has an optimal representation (T, H, η). It follows from (4.3) that, for each strip (J, Z), either

(a) (J, Z) is a spot, or
(b) (J, Z) is a isomorphic to a member of $\mathcal{Z}_0$.

If H is 2-connected, then it follows from (6.3) that G is resolved. Thus, we may assume that H is not 2-connected. Therefore, let $(B_1, B_2, \ldots, B_q)$, with $q \geq 2$, be the block-decomposition of H. Since $q \geq 2$, H has at least two leaf-blocks B, B'. It follows from (6.4) that the strip-block of (H, η) at at least one of these two blocks, B say, is ordinary with respect to G.

First suppose that $|E(B)| = 1$. Let $F \in E(B)$. It follows from (4.5) and (6.6) that the strip (J, Z) of (H, η) at F is either a spot or is isomorphic to a member of one of $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, $\mathcal{Z}_6$, $\mathcal{Z}_9$, $\mathcal{Z}_{10}$, $\mathcal{Z}_{11}$, $\mathcal{Z}_{13}$, or $\mathcal{Z}_{15}$. If (J, Z) is a spot, then the unique vertex in $V(J) \setminus Z$ is a simplicial vertex and the result follows from (2.8). Thus, we may assume that (J, Z) is not a spot. Now, the theorem follows from (6.8), (6.9), (6.10), (6.12), (6.13), (6.15), (6.16), (6.18) and (6.19), respectively. So we may assume that $|E(B)| \geq 2$. It follows from (6.5) that there exists $t \geq 2$ such that $U(B)$ is isomorphic to one of K_2, K_3 K_4, $K_{2,t}$, or $K^+_{2,t}$. Thus, the theorem follows from (6.20) and (6.21). This proves Theorem 1.2. □

Acknowledgments

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Appendix. Illustrations

This appendix contains some figures that graphically illustrate the different types of strips defined in Section 2.3. We have included these figures to give an indication of the structure of the strips, and not to give complete definitions of them. Therefore, in doing so, we aimed at keeping the drawings as simple as possible while still being instructive and, hence, we omitted certain details. The formal definitions can be found in Section 2.3.

For each strip (J, Z), we adopt the convention that end vertices (i.e. the vertices in Z) are drawn as black squares; all other vertices are drawn as black circles. The gray ellipses represent sets of vertices. Strong adjacencies are represented by solid lines, strong antiadjacencies by dashed lines, and semiadjacencies by "wiggly" lines. Parallel solid lines between sets indicate that these sets are strongly complete to each other, and parallel wiggly lines between sets indicate that the adjacency between them is arbitrary (or 'arbitrary' subject to some rules).

Fig. 6 illustrates the hex-expansion construction. Figs. 7–21 illustrate the 15 different types of strips. The strips $\mathcal{Z}_2$, $\mathcal{Z}_8$, $\mathcal{Z}_{11}$ contain complements of matchings between pairs of sets. For obvious reasons, although there are in general lots of adjacencies between these sets, we have only drawn the antiadjacencies between these sets.

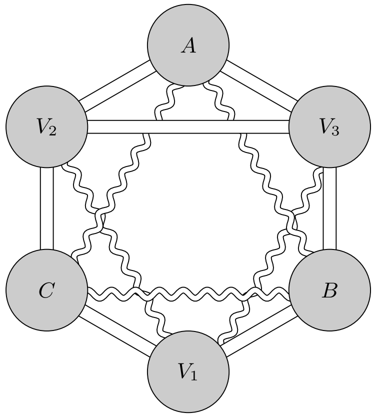

Fig. 6. Schematic drawing of a hex-expansion. The sets A, B, C, V_1, V_2, V_3 are all strong cliques. The double lines indicate which sets are strongly complete to each other, the double curved line indicates which sets have arbitrary adjacencies between them.

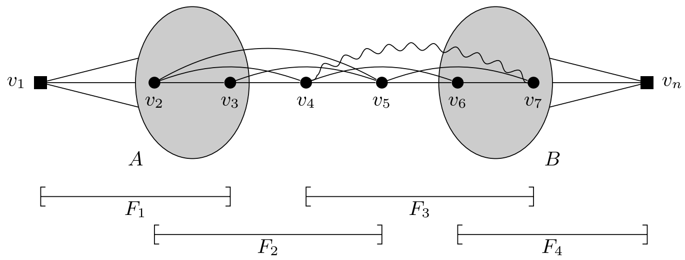

Fig. 7. Schematic drawing of strips in $\mathcal{Z}_1$ (linear interval strips). The sets A and B are strong cliques.

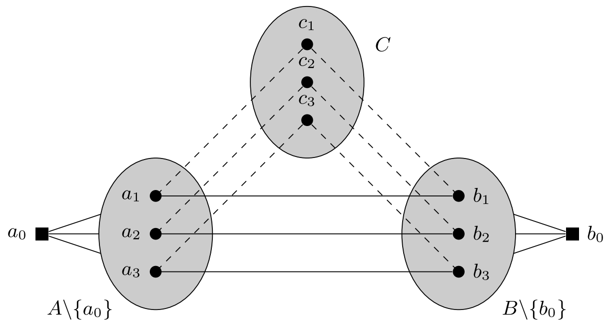

Fig. 8. Schematic drawing of strips in $\mathcal{Z}_2$ (near antiprismatic strips). The sets A, B, C are strong cliques. There is essentially a matching between the sets $A \setminus \{a_0\}$ and $B \setminus \{b_0\}$, and there are essentially complements of matchings between the sets $A \setminus \{a_0\}$ and C and between the sets $B \setminus \{a_0\}$ and C.

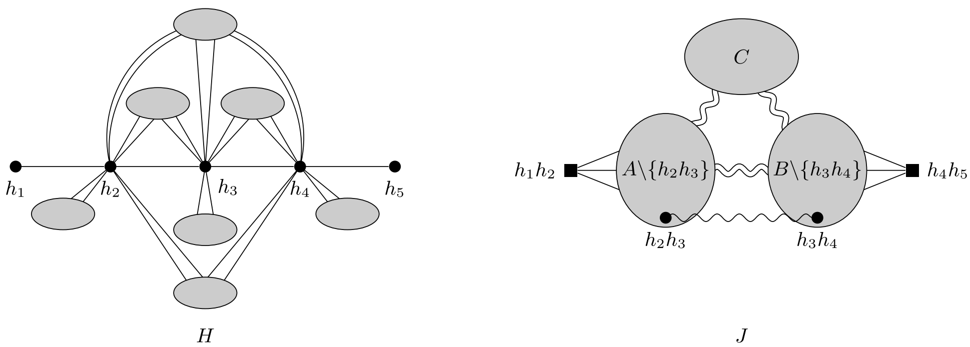

Fig. 9. Schematic drawing of strips in $\mathcal{Z}_3$. On the left: the graph H. On the right: the corresponding strip (J, Z), which is almost a line trigraph of H, with the only exception that h_2h_3 and h_3h_4 are either semiadjacent or strongly antiadjacent. The sets A, B, C satisfy $A = \delta_H(h_2), B = \delta_H(h_4), C = \delta_H(h_3) \setminus \{h_2h_3, h_3h_4)$, where, for $i \in \{2, 3, 4\}$, $\delta_H(h_i)$ is the set of edges in H that are incident with h_i.

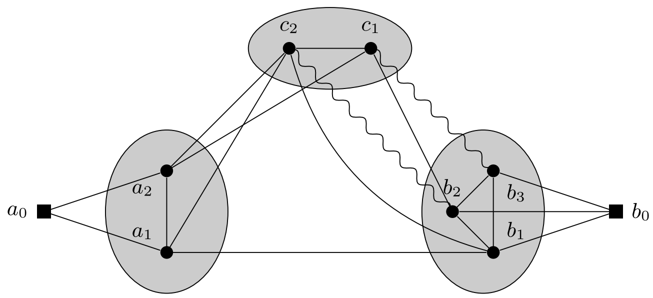

Fig. 10. Schematic drawing of strips in $\mathcal{Z}_4$ (sporadic family of trigraphs of bounded size #1). The pairs b_2, c_2 and b_3, c_1 are semiadjacent.

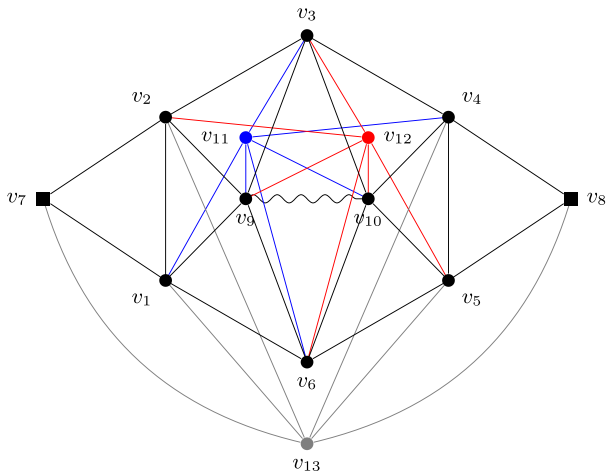

Fig. 11. Schematic drawing of strips in $\mathcal{Z}_5$. The pair v_8v_9 is either semiadjacent or strongly adjacent. The vertices $v_7, v_{11}, v_{12}, v_{13}$ may be deleted.

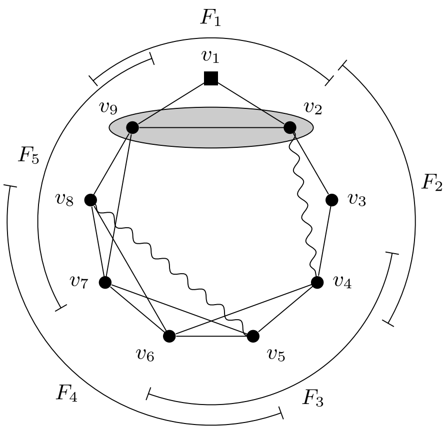

Fig. 12. Example of a strip in $\mathcal{Z}_6$ (long circular interval strips). v_1 is a simplicial vertex. The sets $F_1, \ldots, F_5$ are the intervals.

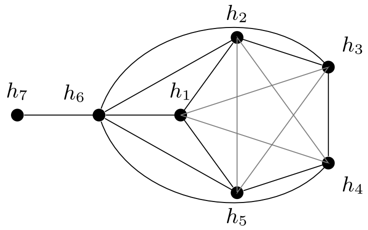

Fig. 13. Schematic drawing of the graph H that underlies strips in $\mathcal{Z}_7$ (modifications of $L(K_6)$). The graph H is the graph shown above, with possibly the edges of any subset of $\{h_1h_3, h_2h_4, h_3h_5, h_2h_4, h_2h_5\}$ deleted (these edges correspond to the five edges 'inside' the 5-cycle h_1-h_2-h_3-h_4-h_5-h_1), and at most two of the edges $\{h_6h_i \mid i = 1, 2, 3, 4, 5\}$ deleted. The trigraph J is essentially the line graph of H (regarded as a trigraph).

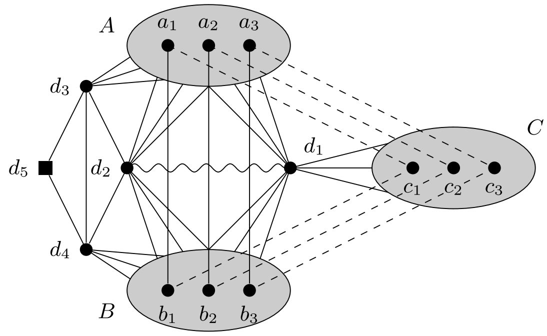

Fig. 14. Schematic drawing of strips in $\mathcal{Z}_8$ (augmented near antiprismatic strips). The pair d_1d_2 is either semiadjacent or strongly adjacent. There is essentially a matching between A and B, and there are complements of matchings between the sets A and C and between the sets B and C.

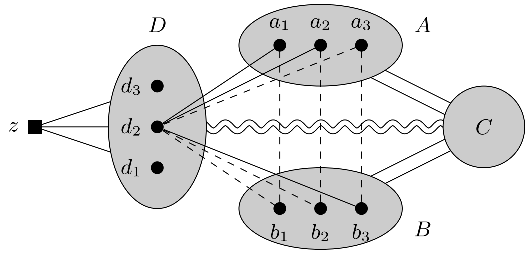

Fig. 15. Schematic drawing of strips in $\mathcal{Z}_9$ (special type of antiprismatic strips). There is essentially a complement of a matching between A and B. For every i and every d, d is strongly adjacent to one of a_i, b_i, and strongly antiadjacent to the other. The adjacency between C and D is arbitrary.

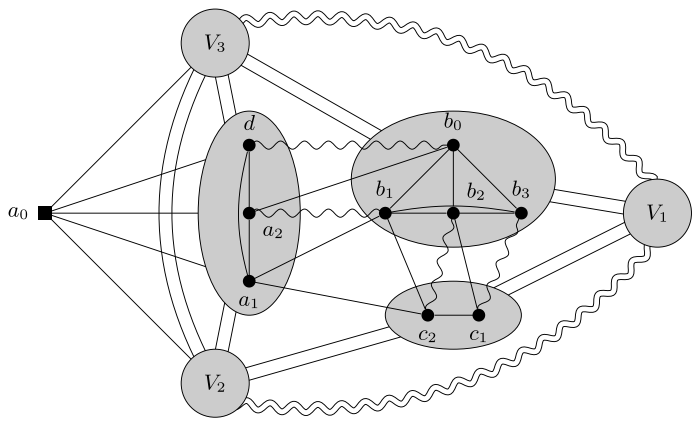

Fig. 16. Schematic drawing of strips in $\mathcal{Z}_{10}$ (sporadic family of trigraphs of bounded size #3). V_2 is strongly complete to V_3, and the adjacency between (V_1, V_2) and (V_1, V_3) is arbitrary. a_2, b_1 are semiadjacent; the pairs b_3, c_1 and b_0, d are either semiadjacent or strongly adjacent.

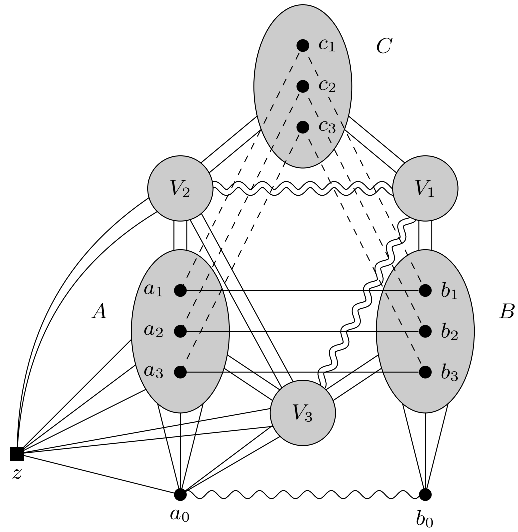

Fig. 17. Schematic drawing of strips in $\mathcal{Z}_{11}$ (hex-expansions of near-antiprismatic trigraphs). There is essentially a matching between A and B, and there are complements of matchings between the sets A and C and between the sets B and C. V_2 is strongly complete to V_3, and the adjacency between the pairs (V_1, V_2) and (V_1, V_3) is arbitrary. a_0, b_0 are either semiadjacent or strongly antiadjacent.

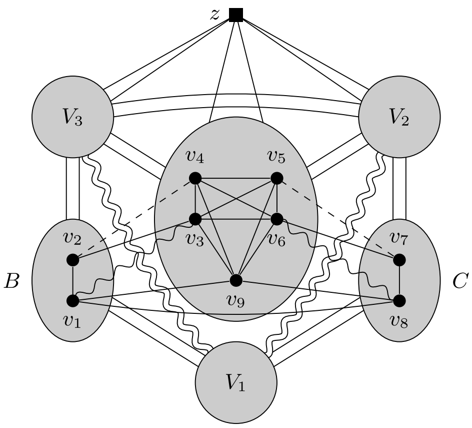

Fig. 18. Schematic drawing of strips in $\mathcal{Z}_{12}$ (hex-expansions of sporadic exception #2). V_1, V_2, V_3 are strong cliques. V_2 is strongly complete to V_3, and the adjacency between the pairs (V_1, V_2) and (V_1, V_3) is arbitrary. The vertices v_3, v_4, v_5, v_6 may be deleted according to some rules.

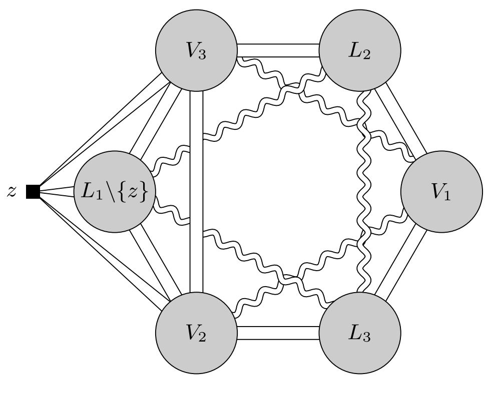

Fig. 19. Schematic drawing of strips in $\mathcal{Z}_{13}$ (hex-expansions of circular interval trigraphs). $L_1, L_2, L_3, V_1, V_2, V_3$ are strong cliques. $L_1 \cup L_2 \cup L_3$ is a circular interval trigraph such that z is strongly anticomplete to $L_2 \cup L_3$. V_2 is strongly complete to V_3, and the adjacency between the pairs (V_1, V_2) and (V_1, V_3) is arbitrary.

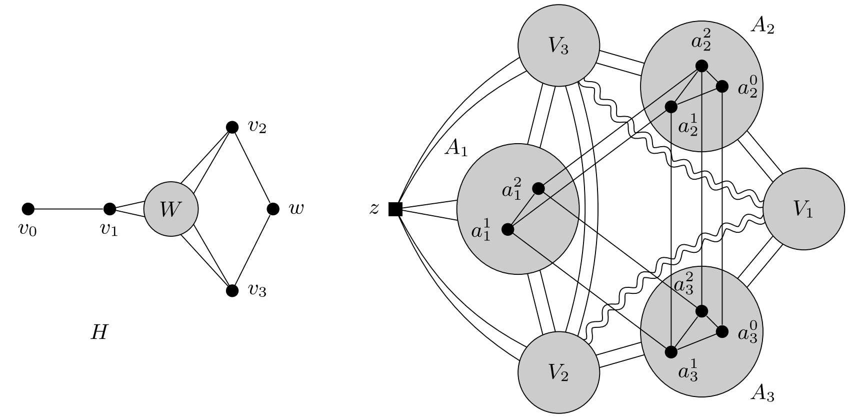

Fig. 20. Schematic drawing of strips in $\mathcal{Z}_{14}$ (hex-expansions of line trigraphs). On the left: the graph H. The set W is a stable set. For $i = 1, 2, 3$, the edges between v_i and $W \cup \{w\}$ correspond to the vertices in A_i. The vertex w may be deleted. On the right: the corresponding hex-expansion of the line trigraph of H. V_2 is strongly complete to V_3, and the adjacency between the pairs (V_1, V_2) and (V_1, V_3) is arbitrary. The vertices a_2^0, a_3^0 are either both present or absent (depending on whether w is present in H).

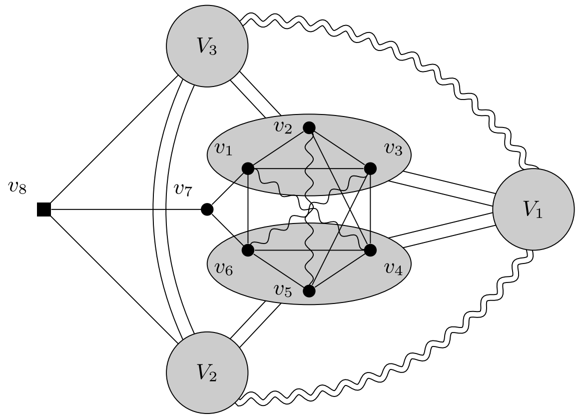

Fig. 21. Schematic drawing of strips in $\mathcal{Z}_{15}$ (hex-expansions of sporadic exception #1). V_2 is strongly complete to V_3, and the adjacency between (V_1, V_2) and (V_1, V_3) is arbitrary. The pairs v_1, v_4 and v_3, v_6 are semiadjacent, and the pair v_2, v_5 is either semiadjacent or strongly antiadjacent.

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